Lecture note 33. Multidimensional Wiener process.

For $0 < \rho < R < \infty$, let $G_{\rho R}$ be the region in $\mathbb{R}^r$ described by

$$G_{\rho R} = \{\boldsymbol{x} \in \mathbb{R}^r \colon \rho < |\boldsymbol{x}| < R\} \tag{33.1}$$

(an annulus between two spheres – or circles, in the two-dimensional case). This region is bounded, so for the first time $\tau = \tau_{\rho R}^{\boldsymbol{x}}$ at which the Wiener process $\boldsymbol{W}_t^{\boldsymbol{x}}$ starting at a point $\boldsymbol{x}$ leaves this region the expectation $E(\tau_{\rho R}^{\boldsymbol{x}}) < \infty$. Let us find the probabilities of the Wiener process leaving the region $G_{\rho R}$ through the smaller sphere (circle) $\{\boldsymbol{x} \colon |\boldsymbol{x}| = \rho\}$:

$$P\{|\boldsymbol{W}_\tau^{\boldsymbol{x}}| = \rho\}. \tag{33.2}$$

This probability can be represented as the expectation

$$E\big(\varphi(\boldsymbol{W}_\tau^{\boldsymbol{x}})\big), \tag{33.3}$$

where φ is a function defined on the boundary $\partial G_{\rho R} = \{\boldsymbol{x} \colon |\boldsymbol{x}| = \rho\} \cup \{\boldsymbol{x} \colon |\boldsymbol{x}| = R\}$ by

$$\varphi(\boldsymbol{x}) = \begin{cases} 1, & |\boldsymbol{x}| = \rho, \\ 0, & |\boldsymbol{x}| = R. \end{cases} \tag{33.4}$$

We know that to find this expectation (this probability) we should solve the Dirichlet problem

$$\begin{aligned} \tfrac{1}{2}\,\Delta u(\boldsymbol{x}) &= 0, & \boldsymbol{x} &\in G_{\rho R}, \\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), & \boldsymbol{x} &\in \partial G_{\rho R}. \end{aligned} \tag{33.5}$$

Solving Problems **28**, **32**, we can write the solution as

$$u(\boldsymbol{x}) = u_{\rho R}(\boldsymbol{x}) = \frac{\ln R - \ln |\boldsymbol{x}|}{\ln R - \ln \rho}, \qquad \rho \le |\boldsymbol{x}| \le R, \tag{33.6}$$

for the dimension $r = 2$, and as

$$u(\boldsymbol{x}) = u_{\rho R}(\boldsymbol{x}) = \frac{|\boldsymbol{x}|^{2-r} - R^{2-r}}{\rho^{2-r} - R^{2-r}}, \qquad \rho \le |\boldsymbol{x}| \le R, \tag{33.7}$$

for $r > 2$.

By Microtheorem 32.2, the probability (33.2) is given by these formulas.

Just as in the case of the one-dimensional Wiener process, let us look what happens as $R \to \infty$ and what as $\rho \to 0^+$.

The limit of a *non-decreasing* family of events A_t is defined as $\bigcup_t A_t$; and the probability of this event is equal to $\lim P(A_t)$ (the limit as t moves to its ultimate bound in the direction in which the family A_t is non-decreasing); the same for a non-increasing family of events, only the limit of the family of events is defined as $\bigcap_t A_t$ in this case.

The family of events

$$A^{\boldsymbol{x}}_{\rho R} = \{|\boldsymbol{X}^{\boldsymbol{x}}_{\tau^{\boldsymbol{x}}_{G_{\rho R}}}| = \rho\} \tag{33.8}$$

(the process leaves $G_{\rho R}$ through the smaller sphere, or circle) is non-decreasing as R grows, for fixed ρ:

$$A^{\boldsymbol{x}}_{\rho R_1} \subseteq A^{\boldsymbol{x}}_{\rho R_2} \tag{33.9}$$

for $R_1 < R_2$. This means that if we left the region $G_{\rho R_1}$ through the smaller sphere (circle), so will we do with the region $G_{\rho R_2}$ (make a picture of the two regions $G_{\rho R_1}$ and $G_{\rho R_2}$, and a trajectory leaving $G_{\rho R_1}$ through the circle of radius ρ).

For fixed R, and ρ *de*creasing towards 0, the family of events $A^{\boldsymbol{x}}_{\rho R}$ is *non-increasing*:

$$A^{\boldsymbol{x}}_{\rho_1 R} \supseteq A^{\boldsymbol{x}}_{\rho_2 R} \tag{33.10}$$

for $\rho_2 < \rho_1$. *This* means that if we left the region $G_{\rho_2 R}$ through the smaller sphere (circle), so will we do with the region $G_{\rho_1 R}$ (again make a picture of the two regions $G_{\rho_1 R}$ and $G_{\rho_2 R}$, and a trajectory leaving $G_{\rho_2 R}$ through the circle of radius ρ_2).

So we have:

$$P\Big(\bigcup_{R>|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R}\Big) = \lim_{R\to\infty} P(A^{\boldsymbol{x}}_{\rho R}), \tag{33.11}$$

$$P\Big(\bigcap_{\rho<|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R}\Big) = \lim_{\rho\to 0^+} P(A^{\boldsymbol{x}}_{\rho R}). \tag{33.12}$$

What is remaining to do is understanding what these unions and intersections are, and evaluating the limits of the function given by (33.6), (33.7).

Let us introduce the notation:

$$\tau_c = \tau^{\boldsymbol{x}}_c = \min\{t\colon\ |\boldsymbol{W}^{\boldsymbol{x}}_t| = c\} \tag{33.13}$$

(as always, taking $\tau^{\boldsymbol{x}}_c = \infty$ if the process never reaches the sphere – or circle – of radius c). We know that $\tau^{\boldsymbol{x}}_c < \infty$ for $c > |\boldsymbol{x}|$, and even that $E(\tau^{\boldsymbol{x}}_c) < \infty$, because this is the first time of leaving the bounded region $\{\boldsymbol{x}:\ |\boldsymbol{x}| < c\}$; we don't know whether $\tau^{\boldsymbol{x}}_c < \infty$ for $c < |\boldsymbol{x}|$ (this is the first time of leaving the *un*bounded region $\{\boldsymbol{x}:\ |\boldsymbol{x}| > c\}$). In these notations, we can rewrite the event (33.8) as

$$A^{\boldsymbol{x}}_{\rho R} = \{\tau^{\boldsymbol{x}}_\rho < \tau^{\boldsymbol{x}}_R\}. \tag{33.14}$$

As $R \to \infty$, we have, obviously,

$$\lim_{R\to\infty} \tau^{\boldsymbol{x}}_R = \infty \tag{33.15}$$

(the Wiener process, being continuous, cannot go to infinity in finite time). This suggests that

$$\bigcup_{R>|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R} = \{\tau^{\boldsymbol{x}}_\rho < \infty\}. \tag{33.16}$$

To prove this, we have to check that $\bigcup_{R>|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R} \subseteq \{\tau^{\boldsymbol{x}}_\rho < \infty\}$ and that $\{\tau^{\boldsymbol{x}}_\rho < \infty\} \subseteq \bigcup_{R>|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R}$.

The first is obvious: if we have reached the sphere (circle) of radius ρ before reaching that of radius R for some R, then we have reached it at *some* time $< \infty$. The second: suppose for an $\omega \in \Omega$ we have $\tau^{\boldsymbol{x}}_\rho(\omega) < \infty$; the function $|\boldsymbol{W}^{\boldsymbol{x}}_t(\omega)|$ is continuous, and so there exists its maximum over a finite closed interval $\max\{|\boldsymbol{W}^{\boldsymbol{x}}_t(\omega)|\colon 0 \le t \le \tau^{\boldsymbol{x}}_\rho(\omega)\}$. For $R > \max\{|\boldsymbol{W}^{\boldsymbol{x}}_t(\omega)|\colon 0 \le t \le \tau^{\boldsymbol{x}}_\rho(\omega)\}$ we have $\tau^{\boldsymbol{x}}_\rho(\omega) < \tau^{\boldsymbol{x}}_R(\omega)$.

So

$$\lim_{R\to\infty} P(A^{\boldsymbol{x}}_{\rho R}) = \lim_{R\to\infty} u_{\rho R}(\boldsymbol{x}) = P\{\tau^{\boldsymbol{x}}_\rho < \infty\}. \tag{33.17}$$

Now to the limit, for fixed R, as $\rho \to 0^+$. Clearly we have

$$\bigcap_{\rho<|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R} = \{\tau^{\boldsymbol{x}}_0 < \tau^{\boldsymbol{x}}_R\}, \tag{33.18}$$

where $\tau^{\boldsymbol{x}}_0$ is the time of reaching the set $\{\boldsymbol{x}\colon |\boldsymbol{x}| = 0\}$, that is, the point $\mathbf{0}$ (check the inclusions $\bigcap_{\rho<|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R} \subseteq \{\tau^{\boldsymbol{x}}_0 < \tau^{\boldsymbol{x}}_R\}$, $\{\tau^{\boldsymbol{x}}_0 < \tau^{\boldsymbol{x}}_R\} \subseteq \bigcap_{\rho<|\boldsymbol{x}|} A^{\boldsymbol{x}}_{\rho R}$ yourselves, making the appropriate picture). So we have:

$$P\{\tau^{\boldsymbol{x}}_0 < \tau^{\boldsymbol{x}}_R\} = \lim_{\rho\to 0^+} P(A^{\boldsymbol{x}}_{\rho R}) = \lim_{\rho\to 0^+} u_{\rho R}(\boldsymbol{x}). \tag{33.19}$$

Now let us evaluate the limits (33.17), (33.19). First, as $R \to \infty$. We have, for the dimension $r = 2$:

$$P\{\tau^{\boldsymbol{x}}_\rho < \infty\} = \lim_{R\to\infty} \frac{\ln R - \ln|\boldsymbol{x}|}{\ln R - \ln\rho} = 1: \tag{33.20}$$

the process almost surely reaches every small circle of a positive radius ρ centered at 0.

If we take circles with an arbitrary center $\boldsymbol{x}_0 \in \mathbb{R}^2$ instead of those centered at the origin $\mathbf{0}$, everything remains the same. So the Wiener trajectory almost surely reaches *every* circle: the trajectory $\boldsymbol{W}^{\boldsymbol{x}}_t$, $t \in [0,\infty)$, covers a dense subset of the plane $\mathbb{R}^2$.

For the dimension $r > 2$:

$$P\{\tau^{\boldsymbol{x}}_\rho < \infty\} = \lim_{R\to\infty} \frac{|\boldsymbol{x}|^{2-r} - R^{2-r}}{\rho^{2-r} - R^{2-r}} = \Big(\frac{\rho}{|\boldsymbol{x}|}\Big)^{r-2}. \tag{33.21}$$

We see that a sphere is reached from the outside with some positive probability that is less than 1.

Now about the limits as $\rho \to 0$. We have, both in the cases $r = 2$ and $r > 2$, for $\boldsymbol{x} \neq 0$:

$$P\{\tau^{\boldsymbol{x}}_0 < \tau^{\boldsymbol{x}}_R\} = \lim_{\rho\to 0^+} P\{\tau^{\boldsymbol{x}}_\rho < \tau^{\boldsymbol{x}}_R\} = \lim_{\rho\to 0^+} u_{\rho R}(\boldsymbol{x}) = 0. \tag{33.22}$$

That is, if we do not start from the point $\mathbf{0}$, almost surely we never reach this point before going out of the sphere (circle) of radius R. Since this is true for *every* $R > |\boldsymbol{x}|$, almost surely we *never* reach $\mathbf{0}$ (note the contrast with the one-dimensional case, where almost surely we reach *every* point of the line).

If we start from the point $\mathbf{0}$ at time $t = 0$, we leave this point (because for every $t > 0$ the random vector $\boldsymbol{W}_t^{\mathbf{0}}$ has a normal distribution – a continuous one, and $P\{\boldsymbol{W}_t^{\mathbf{0}} = \mathbf{0}\} = 0$), and never return to it.

If, instead of considering spheres (circles) centered at $\mathbf{0}$, we take them centered at an arbitrary point $\boldsymbol{x}_0$ (and replace $|\boldsymbol{x}|$ with $|\boldsymbol{x} - \boldsymbol{x}_0|$), we get that for every $\boldsymbol{x}_0 \neq \boldsymbol{x}$

$$P\{\boldsymbol{W}_t^{\boldsymbol{x}} \text{ ever reaches } \boldsymbol{x}_0\} = 0: \tag{33.23}$$

for every point $\boldsymbol{x}_0$ different from our starting point almost surely we never reach it.

Does this mean that almost surely the process $\boldsymbol{W}_t^{\boldsymbol{x}}$ never reaches any point $\boldsymbol{x}_0$ different from $\boldsymbol{x}$ (which seems absurd)?

Let us introduce the event

$$B_{\boldsymbol{x}\boldsymbol{x}_0} = \{\boldsymbol{W}_t^{\boldsymbol{x}} \text{ reaches } \boldsymbol{x}_0 \text{ at some time}\}. \tag{33.24}$$

The question is: we know that $P(B_{\boldsymbol{x}\boldsymbol{x}_0}) = 0$ for every $\boldsymbol{x}_0 \neq \boldsymbol{x}$; does it follow that

$$P\{\boldsymbol{W}_t^{\boldsymbol{x}} \text{ ever reaches any point } \boldsymbol{x}_0 \neq \boldsymbol{x}\} = P\Big(\bigcup_{\boldsymbol{x}_0 \neq \boldsymbol{x}} B_{\boldsymbol{x}\boldsymbol{x}_0}\Big) = 0? \tag{33.25}$$

If the union $\bigcup_{\boldsymbol{x}_0 \neq \boldsymbol{x}} B_{\boldsymbol{x}\boldsymbol{x}_0}$ were a *countable* one, it would follow (by the countable additivity of the probability measure); but the set of all points $\boldsymbol{x}_0 \in \mathbb{R}^r$ that are different from $\boldsymbol{x}$ is clearly *un*countable: so we cannot get (33.25) this way.

But maybe (33.25) still is true for some other reason? It turns out that $P\big(\bigcup_{\boldsymbol{x}_0 \neq \boldsymbol{x}} B_{\boldsymbol{x}\boldsymbol{x}_0}\big) = P\{\boldsymbol{W}_t^{\boldsymbol{x}} \text{ ever reaches any point } \boldsymbol{x}_0 \neq \boldsymbol{x}\} = 1$: because, let me repeat again, at every time $t > 0$ the process $\boldsymbol{W}_t^{\boldsymbol{x}}$ almost surely *is* at some point different from $\boldsymbol{x}$ ($P\{\boldsymbol{W}_t^{\boldsymbol{x}} \neq \boldsymbol{x}\} = 1$).

Finally, we can let $\rho \to 0^+$ in (33.21) and get that

$$\begin{aligned} &P\Big(\bigcap_{\rho < |\boldsymbol{x}|} \{\tau_\rho^{\boldsymbol{x}} < \infty\}\Big) \\ &\quad = P\{\text{the process } \boldsymbol{W}_t^{\boldsymbol{x}} \text{ reaches every sphere of positive radius with center at } \mathbf{0}\} \\ &\quad = \lim_{\rho \to 0^+} \Big(\frac{\rho}{|\boldsymbol{x}|}\Big)^{r-2} = 0. \end{aligned} \tag{33.26}$$

This means that in dimensions $r > 2$ the trajectory $\boldsymbol{W}_t^{\boldsymbol{x}}$ does *not* cover a set that is dense in $\mathbb{R}^r$ (in contrast with the case of $r = 2$).

Another example of application to the Wiener process.

Let us consider $G \subseteq \mathbb{R}^2$ being the upper half-plane:

$$G = \{\boldsymbol{x} = (x^1, x^2) : x^2 > 0\}. \tag{33.27}$$

For a two-dimensional Wiener process $\boldsymbol{W}_t = (W_t^1, W_t^2)$ the time $\tau = \tau_G^{\boldsymbol{x}}$ is almost surely finite – because it is the time at which the one-dimensional Wiener process W_t^2 leaves the half-line $(0, \infty)$. It turns out that we can find the distribution of the point $\boldsymbol{W}_\tau^{\boldsymbol{x}}$ at which the two-dimensional Wiener process leaves the half-plane G for the first time. Of course, the second coordinate of this point is 0 (it lies on the horizontal coordinate axis $x^2 = 0$); so it is, in fact, about the distribution of the random variable $W_\tau^1 = W^{1,\,x^1}_{\tau_G^{\boldsymbol{x}}}$.

According to Problems **33**, **34** and Microtheorem 32.2, we have for every bounded continuous function $\varphi(y)$, $-\infty < y < \infty$:

$$E\bigl(\varphi(W^{1,\,x^1}_{\tau_G^{\boldsymbol{x}}})\bigr) = \int_{-\infty}^{\infty} \frac{\pi^{-1} \cdot x^2}{(x^2)^2 + (x^1 - y)^2} \cdot \varphi(y)\, dy. \tag{33.28}$$

This means that the distribution of the random variable $W^{1,\,x^1}_{\tau_G^{\boldsymbol{x}}}$ for the Wiener process starting from the point $\boldsymbol{x} = (x^1, x^2)$, $-\infty < x^1 < \infty$, $x^2 > 0$, is a continuous one, with density

$$p_{x^1,\,x^2}(y) = \frac{\pi^{-1} \cdot x^2}{(x^2)^2 + (y - x^1)^2}. \tag{33.29}$$

This distribution is one of the family of *Cauchy distributions* (the most frequently mentioned member of this family is the Cauchy distribution with parameters $x^1 = 0$, $x^2 = 1$, with density $p(y) = \dfrac{\pi^{-1}}{1 + y^2}$).

I have to formulate here several theorems about characterization of distributions, in particular, by expectations of continuous functions of random variables. These theorems do not belong to the theory of stochastic equations, or of stochastic processes, but rather to plain probability theory.

Theorem 33.1. *Suppose X and Y are two random variables. If they have the same (cumulative) distribution function:*

$$F_X(t) = P\{X \le t\} = F_Y(t) = P\{Y \le t\}, \quad -\infty < t < \infty, \tag{33.30}$$

then their distributions coincide: $\mu_X = \mu_Y$ – which means that for every set $C \subseteq \mathbb{R}$ (to be precise, for every set for which we can guarantee that $\{\omega\colon\ X(\omega) \in C\}$, $\{\omega\colon\ Y(\omega) \in C\}$ are events, i. e. belong to our basic class $\mathcal{F}$)

$$\mu_X(C) = P\{X \in C\} = \mu_Y(C) = P\{Y \in C\}. \tag{33.31}$$

This theorem is usually stated (or just taken as something so obvious that it does not even require any statement about it) in the elementary course of probability theory; but its accurate *formulation* and **proof** require knowing something about measure theory (and not only about its simpler results, such as, e. g., monotone-convergence or dominated-convergence theorem). I am not giving the proof.

Theorem 33.2. *Suppose X and Y are two random variables. If for every bounded continuous function $\varphi(x)$ we have*

$$E\bigl(\varphi(X)\bigr) = E\bigl(\varphi(Y)\bigr), \tag{33.32}$$

then their distributions coincide: $\mu_X = \mu_Y$.

Proof (taking Theorem 33.1 as already established). For every $t \in (-\infty, \infty)$ and every $\varepsilon > 0$, let us consider the bounded continuous function $\varphi_{t,\,\varepsilon}$:

$$\varphi_{t,\,\varepsilon}(x) = \begin{cases} 1, & x \le t, \\ \dfrac{t+\varepsilon-x}{\varepsilon}, & t \le x \le t+\varepsilon, \\ 0, & x \ge t+\varepsilon \end{cases} \tag{33.33}$$

(draw the graph of this function – consisting of three segments of straight line; for $x = t$ and for $x = t+\varepsilon$ the two formulas by which $\varphi_{t,\,\varepsilon}(x)$ is defined yield the same result; look at your picture and think about how to prove that this function is continuous). (Bounded, too.)

By (33.32) we have:

$$E\big(\varphi_{t,\,\varepsilon}(X)\big) = E\big(\varphi_{t,\,\varepsilon}(Y)\big). \tag{33.34}$$

For every $x \in (-\infty, \infty)$ we have:

$$\lim_{\varepsilon \to 0^+} \varphi_{t,\,\varepsilon}(x) = I_{(-\infty,\,t]}(x) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0 \end{cases} \tag{33.35}$$

(the limit is a *dis*continuous function): for $x \le t$ because all pre-limit quantities are equal to 1; for $x > t$ because the pre-limit quantity is equal to 0 for $\varepsilon < x - t$. So by the dominated-convergence theorem we have:

$$\begin{aligned} F_X(t) = P\{X \le t\} &= E\big(I_{(-\infty,\,t]}(X)\big) = E\big(\lim_{\varepsilon \to 0^+} \varphi_{t,\,\varepsilon}(X)\big) = \lim_{\varepsilon \to 0^+} E\big(\varphi_{t,\,\varepsilon}(X)\big) \\ &= \lim_{\varepsilon \to 0^+} E\big(\varphi_{t,\,\varepsilon}(Y)\big) = E\big(\lim_{\varepsilon \to 0^+} \varphi_{t,\,\varepsilon}(Y)\big) = E\big(I_{(-\infty,\,t]}(Y)\big) = F_Y(t). \end{aligned} \tag{33.36}$$

Theorem 33.3. *Suppose* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are two* r*-dimensional random vectors. If for every bounded continuous function* $\varphi(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^r$, *we have*

$$E\big(\varphi(X)\big) = E\big(\varphi(Y)\big), \tag{33.37}$$

then their distributions coincide: $\mu_{\boldsymbol{X}} = \mu_{\boldsymbol{Y}}$.

The **proof** is essentially the same: using the multidimensional distribution functions.

This result is true also for continuous functions on some subset $A \subset \mathbb{R}^r$, say, on some surface in this space – if the random variables (vectors) $\boldsymbol{X}$, $\boldsymbol{Y}$ take only values in the set A:

Theorem 33.4. *Suppose* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *take values in some set* $A \subset \mathbb{R}^r$. *If for every bounded continuous function* $\varphi(\boldsymbol{x})$, $\boldsymbol{x} \in A$, *equality* (33.37) *holds, then their distributions coincide*: $\mu_{\boldsymbol{X}} = \mu_{\boldsymbol{Y}}$.

It is proved in the theory of elliptic differential equations, under some conditions, that for every bounded continuous function $\varphi(\boldsymbol{x})$ on the boundary ∂G of our region G the solution of the Dirichlet problem

$$\begin{aligned} Lu(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial G, \end{aligned} \tag{33.38}$$

(oh, I am speaking about *the* solution as if it were obvious that it is unique; it *is* unique for bounded regions, and it is unique in the class of bounded functions if G is unbounded, but $\tau_G^{\boldsymbol{x}} < \infty$ almost surely) has the following integral representation:

$$u(\boldsymbol{x}) = \int_{\partial G} p(\boldsymbol{x},\, \boldsymbol{y}) \cdot \varphi(\boldsymbol{y})\ S(d\boldsymbol{y}), \tag{33.39}$$

where $S(d\boldsymbol{y})$ denotes integration with respect to the surface area on ∂G (for $r = 2$, the boundary ∂G is not a *surface*, but rather a curve, and the integral is taken not with respect to the surface area, but with respect to the *length*, the good notations for which would be with the differential $\ell(d\boldsymbol{y})$; but I don't want to complicate the formulation by mentioning that for $r = 2$ the integral is taken with respect to the curve length, for $r = 3$, with respect to the surface area, for $r = 4$ with respect to the three-dimensional volume on a three-dimensional *hypersurface*, etc. – better we formulate it like this: with respect to the surface measure on a surface of the appropriate dimension). The function $p(\boldsymbol{x}, \boldsymbol{y})$ is called *the Poisson kernel* (so: the Poisson kernel for G being the upper half-plane is given by (33.29); in fact, what Poisson invented was only for $L = \Delta$, the Laplace operator – but we can use the same name in a more general situation).

Every time that this result is established in the theory of PDEs, we can understand the probabilistic meaning of the Poisson kernel $p(\boldsymbol{x}, \boldsymbol{y})$: as a function of $\boldsymbol{y}$, it is the probability density of the random point $\boldsymbol{X}^{\boldsymbol{x}}_{\tau_G^{\boldsymbol{x}}}$ on the surface ∂G (the density with respect to the surface measure: with integration in the equality $P\{\boldsymbol{X}^{\boldsymbol{x}}_{\tau_G^{\boldsymbol{x}}} \in C\} = \int_C p(\boldsymbol{x}, \boldsymbol{y})\ S(d\boldsymbol{y})$ with respect to the surface measure). The first argument $\boldsymbol{x}$ of the Poisson kernel is not its argument in which it serves as a density: it is the initial point from which our diffusion process starts at zero time.

So: look up a book on partial differential equations, and find the formula for the Poisson kernel for some region G: you have a formula for the distribution density on the boundary of the point at which our process leaves our region.