Lecture note 34. Elliptic equations with the term $c(\boldsymbol{x}) \cdot u(\boldsymbol{x})$.

Now let us speak of elliptic equations of the form

$$Lu(\boldsymbol{x}) + c(\boldsymbol{x}) \cdot u(\boldsymbol{x}) = \frac{1}{2} \sum_{i,\,j=1}^{r} a_{ij}(\boldsymbol{x}) \cdot \frac{\partial^2 u}{\partial x^i \partial x^j}(\boldsymbol{x}) + \sum_{i=1}^{r} b_i(\boldsymbol{x}) \cdot \frac{\partial u}{\partial x^i}(\boldsymbol{x}) + c(\boldsymbol{x}) \cdot u(\boldsymbol{x}) = g(\boldsymbol{x}). \tag{34.1}$$

At first, let us suppose that the functions $u(\boldsymbol{x})$ and $c(\boldsymbol{x})$ are defined in the whole space $\mathbb{R}^r$.

Let Z_t, $t > 0$, be the random function defined by

$$Z_t = \int_0^t c(\boldsymbol{X}_s)\; ds. \tag{34.2}$$

We have:

$$\begin{aligned} dZ_t &= c(\boldsymbol{X}_t)\, dt, \\ dX_t^i &= \sum_{k=1}^{n} \sigma_{ik}(\boldsymbol{X}_t)\, dW_t^k + b_i(\boldsymbol{X}_t)\, dt, \quad i = 1, ..., r. \end{aligned} \tag{34.3}$$

By Itô's formula for the function $F(z,\, \boldsymbol{x}) = e^z \cdot u(\boldsymbol{x})$,

$$d\big(e^{Z_t} \cdot u(\boldsymbol{X}_t)\big) = e^{Z_t} \sum_{i=1}^{r} \sum_{k=1}^{n} \frac{\partial u}{\partial x^i}(\boldsymbol{X}_t)\, \sigma_{ik}(\boldsymbol{X}_t)\, dW_t^k + e^{Z_t} [Lu(\boldsymbol{X}_t) + c(\boldsymbol{X}_t) \cdot u(\boldsymbol{X}_t)]\, dt \tag{34.4}$$

(I entrust to you checking the calculations).

Let us define the random function Y_t by

$$Y_t = \exp\Big\{ \int_0^t c(\boldsymbol{X}_s)\; ds \Big\} \cdot u(\boldsymbol{X}_t) - \int_0^t \exp\Big\{ \int_0^s c(\boldsymbol{X}_v)\; dv \Big\} \cdot [Lu(\boldsymbol{X}_s) + c(\boldsymbol{X}_s) \cdot u(\boldsymbol{X}_s)]\, ds. \tag{34.5}$$

Because of the equality (34.4), this difference is a stochastic integral, and as such, a martingale.

Considering this martingale at the random times $\min(\tau,\, t_*)$ and using our Microtheorem 29.1, we obtain probabilistic representations for solutions of Dirichlet problems for the equation (34.1).

Of course, there are at least as many microtheorems to formulate here and at least as many examples to give as for the case $c \equiv 0$; no time for all of this, let us consider at least something.

Let us restrict ourselves to the one-dimensional case, to the Wiener process as the diffusion process under consideration, and to $c(x) \equiv c = \text{const}$. The expectation $w(x) = E(e^{c\tau^x_{(a,\,b)}})$ is the same as the solution of the Dirichlet problem

$$\begin{aligned} &\tfrac{1}{2}\, w''(x) + c \cdot w(x) = 0, \qquad a < x < b, \\ &w(a) = w(b) = 1 \quad - \end{aligned} \tag{34.6}$$

provided this expectation is finite.

Indeed, suppose the function $w(x)$, $a \le x \le b$, is a solution of (34.6), and can be extended outside the interval $[a, b]$ so as to be smooth and bounded with its derivatives. The random function

$$Y_t = e^{ct} w(W_t^x) - \int_0^t e^{cs} [\tfrac{1}{2}\, w''(W_s^x) + c \cdot w(W_s^x)]\, ds \tag{34.7}$$

is a martingale; so

$$E(Y_{\min(\tau^x_{(a,\,b)},\, t_*)}) = E(Y_0), \tag{34.8}$$

$$E\Big(e^{c\cdot\min(\tau^x_{(a,\,b)},\, t_*)} w(W^x_{\min(\tau^x_{(a,\,b)},\, t_*)}) - \int_0^{\min(\tau^x_{(a,\,b)},\, t_*)} e^{cs} [\tfrac{1}{2}\, w''(W_s^x) + c \cdot w(W_s^x)]\, ds\Big) = w(x). \tag{34.9}$$

The integral here is equal to 0, because $\frac{1}{2}\, w'' + cw = 0$ in the interval (a, b), so only the term $e^{c\cdot\min(\tau,\, t_*)} w(W^x_{\min(\tau,\, t_*)})$ remains. Now we take $t_* \to \infty$.

For $c \le 0$, all random variables $e^{c\cdot\min(\tau,\, t_*)} w(W^x_{\min(\tau,\, t_*)})$ are dominated in absolute value by the constant $\sup_{x\in[a,\,b]} |w(x)|$; for $c > 0$, by the random variable $\sup_{x\in[a,\,b]} |w(x)| \cdot e^{c\tau^x_{(a,\,b)}}$. In both cases, the dominating random variables have finite expectation; so by the dominated-convergence theorem we have:

$$\begin{aligned} E(e^{c\tau^x_{(a,\,b)}}) &= E\big(e^{c\tau^x_{(a,\,b)}} \cdot w(W^x_{\tau^x_{(a,\,b)}})\big) = E\big(\lim_{t_*\to\infty} e^{c\cdot\min(\tau,\, t_*)} w(W^x_{\min(\tau,\, t_*)})\big) \\ &= \lim_{t_*\to\infty} E\big(e^{c\cdot\min(\tau,\, t_*)} w(W^x_{\min(\tau,\, t_*)})\big) = \lim_{t_*\to\infty} w(x) = w(x). \end{aligned} \tag{34.10}$$

Now to solving the boundary-value problem (34.6).

For $c < 0$, the general solution of the equation, not taking into account the boundary conditions, is

$$C_1\, e^{\sqrt{-2c}\cdot x} + C_2\, e^{-\sqrt{-2c}\cdot x}; \tag{34.11}$$

Using the boundary conditions, we get a system of two linear algebraic equations for C_1, C_2; solving it, we get:

$$C_1 = \frac{e^{-\sqrt{-2c}\cdot a} - e^{-\sqrt{-2c}\cdot b}}{e^{\sqrt{-2c}\cdot(b-a)} - e^{-\sqrt{-2c}\cdot(b-a)}}, \qquad C_2 = \frac{e^{\sqrt{-2c}\cdot b} - e^{\sqrt{-2c}\cdot a}}{e^{\sqrt{-2c}\cdot(b-a)} - e^{-\sqrt{-2c}\cdot(b-a)}}, \tag{34.12}$$

$$w(x) = \frac{\sinh\big(\sqrt{-2c}\cdot(x-a)\big) + \sinh\big(\sqrt{-2c}\cdot(b-x)\big)}{\sinh\big(\sqrt{-2c}\cdot(b-a)\big)}, \qquad a \le x \le b \tag{34.13}$$

(with hyperbolic sines). As for $E(e^{c\tau^x_{(a,\,b)}})$ being finite, for negative c this question does not arise. So the expectation $E(e^{c\tau^x_{(a,\,b)}})$ is given by formula (34.13).

For $c = 0$ there is nothing to solve; for $c > 0$ the solution $w(x)$ exists for $\sqrt{2c}\cdot(b-a) \neq \pi, 2\pi, 3\pi, \ldots$, and is given by

$$w(x) = \frac{\sin\big(\sqrt{2c}\cdot(x-a)\big) + \sin\big(\sqrt{2c}\cdot(b-x)\big)}{\sin\big(\sqrt{2c}\cdot(b-a)\big)}, \qquad a \le x \le b \tag{34.14}$$

(with *trigonometric* sines; check it). But *is* the expectation $E(e^{c\tau^x_{(a,\,b)}})$ finite?? If it is, it is given by formula (34.14); but is it?

Let us consider c such that $\sqrt{2c}\cdot(b-a) < \pi$ (that is, $0 < c < \pi^2/2(b-a)^2$). For these c the function $w(x)$ defined by (34.14) is positive in the interval $[a, b]$; so the random variables

$$Y_{\min(\tau^x_{(a,\,b)})} = e^{c\cdot\min(\tau,\,t_*)} w(W^x_{\min(\tau,\,t_*)}) \tag{34.15}$$

are all positive. We cannot use the dominated-convergence theorem: the dominating random variable is $\max_{x\in[a,\,b]} w(x)\cdot e^{c\tau}$, and our problem is that *we don't know* whether $E(e^{c\tau})$ is finite. We cannot use the monotone-convergence theorem either: of course $e^{c\cdot\min(\tau,\,t_*)}$ does not decrease as t_* grows, but we cannot say the same about the second factor, $w(W^x_{\min(\tau,\,t_*)})$ is not such. Luckily, our theory of limit passage for random variables is held by *three* pillars: Theorems 2.1, 2.2, and 2.3 (the Fatou's Lemma). By Fatou's Lemma we have:

$$\begin{aligned} E(e^{c\tau^x_{(a,\,b)}}) &= E\big(\lim_{t_*\to\infty} e^{c\cdot\min(\tau,\,t_*)} w(W^x_{\min(\tau,\,t_*)})\big) = E\big(\varliminf_{t_*\to\infty} e^{c\cdot\min(\tau,\,t_*)} w(W^x_{\min(\tau,\,t_*)})\big) \\ &\le \varliminf_{t_*\to\infty} E\big(e^{c\cdot\min(\tau,\,t_*)} w(W^x_{\min(\tau,\,t_*)})\big) = \varliminf_{t_*\to\infty} w(x) = w(x) < \infty. \end{aligned} \tag{34.16}$$

So for $c < \pi^2/2(b-a)^2$ our expectation *is* finite, and so it is given by the formula we have written: in fact, not only $E(e^{c\tau^x_{(a,\,b)}}) \le w(x)$ (which we deduced using Fatou's Lemma), but also $E(e^{c\tau^x_{(a,\,b)}}) = w(x)$ (which is obtained, given the inequality, using the dominated-convergence theorem).

What will be for $c \ge \pi^2/2(b-a)^2$? For $c = c_0 = \pi^2/2(b-a)^2$ we have by the monotone-convergence theorem:

$$E(e^{c_0\tau^x_{(a,\,b)}}) = \lim_{c\to c_0^-} E(e^{c\tau^x_{(a,\,b)}}) = \lim_{c\to c_0^-} \frac{\sin\big(\sqrt{2c}\cdot(x-a)\big) + \sin\big(\sqrt{2c}\cdot(b-x)\big)}{\sin\big(\sqrt{2c}\cdot(b-a)\big)}. \tag{34.17}$$

For $a < x < b$ the numerator has a positive limit, namely, $\sin\big(\sqrt{2c_0}\cdot(x-a)\big) + \sin\big(\sqrt{2c_0}\cdot(b-x)\big)$, and the limit of the denominator (which is positive) is $\sin\big(\sqrt{2c_0}\cdot(b-a)\big) = 0$. So we have:

$$E(e^{c_0\tau^x_{(a,\,b)}}) = \infty. \tag{34.18}$$

For $c > c_0$ it is obvious that $e^{c\tau^x_{(a,\,b)}} > e^{c_0\tau^x_{(a,\,b)}}$, and the expectation $E(e^{c\tau^x_{(a,\,b)}})$ is also infinite.

So: the expectation $E(e^{c\tau^x_{(a,\,b)}})$ is infinite for $c \geq c_0$, and it is finite and given by formulas (34.13), (34.14) for $c < c_0$.

By the way: the function

$$M_{\tau^x}(c) = E(e^{c \cdot \tau^x}) \tag{34.19}$$

is the moment-generating function of the random variable τ^x. Wee know that the moment-generating function of a random variable determines uniquely the distribution of this random variable – if only it is finite in some interval consisting of more than one point. Of course, the function (34.19) is finite at least on the left half-line: $c \in (-\infty, 0]$; so, *in principle*, we can find the distribution of the exit time τ^x by solving partial differential equations. It may be difficult; but possible.