MATH 657 STOCHASTIC DIFFERENTIAL EQUATIONS SPRING 2007

Lecture note 35. More about Markov processes. The strong Markov property.

The definition of a Markov process *with transition function* $\mu_{t,\,x,\,s}(C)$, $t,\,s \in T$, $t \le s$, $x \in \mathrm{Sp}$, $C \subseteq \mathrm{Sp}$, was written like this: for $t < s$

$$P\{X_s \in C \| X_u,\, u \le t\} = \mu_{t,\,X_t,\,s}(C). \tag{35.1}$$

For a time-homogeneous process, i. e., with the transition function $\mu_{t,\,x,\,s}(C)$ depending on the difference $s - t$ rather than on t and s separately:

$$\mu_{t,\,x,\,s}(C) = \mu_{s-t,\,x}(C), \tag{35.2}$$

formula (35.1) can be rewritten as

$$P\{X_{t+h} \in C \| X_u,\, u \le t\} = \mu_{h,\,X_t}(C) \tag{35.3}$$

for $h > 0$.

Is the same true if we replace the time t with some *random variable* τ? Is

$$P\{X_{\tau+h} \in C \| X_u,\, u \le \tau\} = \mu_{h,\,X_\tau}(C) \tag{35.4}$$

(of course this equality only makes sense for ω's for which $\tau(\omega) < \infty$)?

If τ is not a stopping time, definitely **no**. Say τ is the last time that the Wiener process with negative drift added $X_t = W_t + bt$ is ≥ 0 (the drift $b < 0$; it can be proved that almost surely $\lim_{t\to\infty}(W_t + bt) = -\infty$, and so the last time τ in the positive half-line is finite). Clearly we have: $X_\tau = 0$, $X_{\tau+h} < 0$, and so

$$P\{X_{\tau+h} \in [0,\infty) \| X_u,\, u \le \tau\} = 0, \tag{35.5}$$

while

$$\mu_{h,\,X_\tau}[0,\infty) = \mu_{h,\,0}[0,\infty) = \int_0^\infty \frac{1}{\sqrt{2\pi h}}\, e^{-(x-bh)^2/2h}\, dy > 0. \tag{35.6}$$

If τ is a stopping time, (35.4) seems plausible. However not for every Markov process X_t and not for every stopping time does it hold.

Example 35.1. Let $W_t^{x_0}$ denote the one-dimensional Wiener process starting from the point x_0 at time 0. Let us construct our process $X_t^{x_0}$, $t \ge 0$, starting from an arbitrary point $x_0 \in \mathbb{R}$, so:

$$X_t^{x_0} = \begin{cases} W_t^{x_0} & \text{if } x_0 \ne 0, \\ 0 & \text{if } x_0 = 0. \end{cases} \tag{35.7}$$

Draw a picture of a trajectory starting at a point $x_0 \ne 0$ and hitting 0 some number of times (as the case will be almost surely), and of a trajectory starting at the point 0 (not much to draw here).

A question: And when $X_t^{x_0}$ hits 0, it does not stick to this point and stop? – An interesting question, meaning, in fact: "Do you really mean what you wrote (formula (35.7)), or you mean something else?" Answer: yes, I did mean what I formulated. The process $X_t^{x_0}$ does *not* stick at the point 0 however much our gut feeling is telling us that it should (but we don't understand why we have this feeling): according to the definition (35.7), to which I am going to stick, the process $X_t^{x_0}$ passes the point 0 taking no notice of the fact. So let us look at what happens if we stick to this definition.

The transition function $\mu_{h,\,x}(C)$ of this process can be found (for $h > 0$) as

$$\mu_{h,\,x}(C) = P\{X_h^x \in C\} = \begin{cases} \displaystyle\int_C \frac{1}{\sqrt{2\pi h}}\, e^{-(y-x)^2/2h}\, dy, & x \neq 0, \\ I_C(0), & x = 0. \end{cases} \tag{35.8}$$

Let us check that the process $X_t^{x_0}$ is, for every x_0, a Markov process with the transition function (35.8).

For $x_0 = 0$ we have: $X_t^0 \equiv 0$, and

$$P\{X_{t+h}^0 \in C \| X_u^0,\, 0 \le u \le t\} = \begin{Bmatrix} 1 & \text{if } C \ni 0 \\ 0 & \text{if } C \not\ni 0 \end{Bmatrix} = \mu_{h,\,0}(C). \tag{35.9}$$

For $x_0 \neq 0$, the distribution of $X_t^{x_0}$ is a continuous one (a normal one), and

$$P\{X_t^{x_0} = 0\} = \int_0^0 \frac{1}{\sqrt{2\pi t}}\, e^{-(y-x_0)^2/2t}\, dy = 0, \tag{35.10}$$

for every $t \ge 0$ almost surely $X_t^{x_0} \neq 0$. So almost surely

$$\begin{aligned} P\{X_{t+h}^{x_0} \in C \| X_u^{x_0},\, 0 \le u \le t\} &= P\{W_{t+h}^{x_0} \in C \| W_u^{x_0},\, 0 \le u \le t\} \\ &= \int_C \frac{1}{\sqrt{2\pi t}}\, e^{-(y-X_t^{x_0})^2/2h}\, dy = \mu_{h,\,X_t^{x_0}}(C). \end{aligned} \tag{35.11}$$

But for the stopping time τ at which our process (having continuous trajectories) first reaches 0 we have, for $x_0 \neq 0$, $h > 0$, and $C = \{0\}$:

$$P\{X_{\tau+h}^{x_0} \in C \| X_u^{x_0},\, u \le \tau\} = P\{W_{\tau+h}^{x_0} = 0 \| W_u^{x_0},\, u \le \tau\} = 0 \neq \mu_{h,\,0}\{0\} = 1. \tag{35.12}$$

So we should introduce *a class* of Markov processes for which (35.4) holds for every stopping time, and some name for the property (35.4). The property (35.4) is called *the strong Markov property*, and the class of Markov processes for which it holds for every stopping time, the class of *strong Markov processes*. We see that at least the Markov process (35.7) is not a strong Markov one. The strong Markov property was used, without proper justification, especially in papers at a "physicists' level of rigorousness", before anybody understood what this is about and if any justification was needed. The credit for first understanding this and for studying the class of strong Markov processes goes to E.Dynkin and A.Yushkevich. It turned out that this class is a pretty wide one – but it is not the class of *all* Markov processes.

Before we go further, let us make precise what we mean by the equality (35.4) for ω's for which $\tau(\omega) = \infty$ (which is possible for stopping times, in some cases with positive probability). We take that for such ω's the event $\{X_{\tau+h} \in C\}$ (which would mean that $X_\infty \in C$, making no sense) does *not* occur; this takes care of the precise meaning of the event under the sign of conditional probability in the left-hand side of (35.4).

But in the right-hand side we have $\mu_{h,\,X_\tau}(C)$, which definitely makes no sense for $\tau = \infty$, again because X_∞ makes no sense. So we replace the expression $\mu_{h,\,X_\tau}(C)$, making no sense, with 0. So the precise formulation of the strong Markov property for stopping times taking the value ∞ with nonzero probability is:

$$P\{X_{\tau+h} \in C \| X_u,\, u \le \tau\} = \begin{cases} \mu_{h,\,X_\tau}(C) & \text{if } \tau < \infty, \\ 0 & \text{if } \tau = \infty. \end{cases} \tag{35.13}$$

We can reformulate the strong Markov property (35.13) (as well as the Markov property (35.3)) in the language of expectations: for every bounded function $f(x)$, $x \in \mathrm{Sp}$,

$$E\bigl(f(X_{\tau+h}) \| X_u,\, u \le \tau\bigr) = \begin{cases} E\bigl(f(X_h^x)\bigr)\big|_{x=X_t} & \text{if } \tau < \infty, \\ 0 & \text{if } \tau = \infty, \end{cases} \tag{35.14}$$

where $f(X_{\tau+h})$ is replaced with 0 if $\tau = \infty$, and X_t^x is the Markov process starting from the point $x \in \mathrm{Sp}$ at time $t = 0$ with the same transition function $\mu_{h,\,x}(C)$ as the process X_t.

Note that in the right-hand side we have X^x, the process that starts at time 0 from a point $x \in \mathrm{Sp}$ (which is subsequently replaced with the random variable X_τ), while in the left-hand side it is just X. This is because we don't assume that our basic Markov process X_t starts from a *non-random* point at $t = 0$; and even if it does, the initial point, say, x_0, needn't be the same as the initial (starting) point x in the right-hand side (which is, let us say it again, subsequently replaced with a random variable).

Formula (35.13) is a particular case of (35.14) for $f(x) = I_C(x)$. Deducing (35.14) from (35.13) requires only approximation of an arbitrary bounded function with functions taking finitely many values (I am skipping the proof).

By Theorem 34.4, it is enough to postulate that (35.14) is satisfied only for bounded *continuous* functions $f(x)$.

Now several results about the strong Markov property.

Microtheorem 35.1. *Let X_t be a time-homogeneous Markov process*; *let τ be a stopping time* (with respect to X_t, of course) *taking countably many values $t_1 < t_2 < \ldots < t_k < \ldots$, and possibly the value ∞.*

Then almost surely (35.13) *holds.*