

Lecture note 39. Controlled diffusion processes, continued.

Theorem 38.1. *Let G be a region in \mathbb{R}^r . Suppose a function $v(\mathbf{x})$, $\mathbf{x} \in G \cup \partial G$ is a solution of the nonlinear Dirichlet problem*

$$\begin{aligned} \mathfrak{L}v(\mathbf{x}) &= 0, & \mathbf{x} \in G, \\ v(\mathbf{x}) &= \varphi(\mathbf{x}), & \mathbf{x} \in \partial G, \end{aligned} \tag{38.17}$$

and suppose that this function can be extended to the whole \mathbb{R}^r to form a function that is twice continuously differentiable and bounded together with its first and second derivatives.

Let $\hat{u}(\mathbf{x})$ be the value (or one of such values) at which $\max_{u \in U} L_u v(\mathbf{x})$ is reached: $\mathfrak{L}v(\mathbf{x}) = L_{\hat{u}(\mathbf{x})}v(\mathbf{x})$ for $\mathbf{x} \in G$, and let $\hat{u}(\mathbf{x})$ be defined arbitrarily for $\mathbf{x} \in \mathbb{R}^r \setminus G$. Let us consider the control strategy

$$\hat{u}(t; \mathbf{x}_s, 0 \leq s \leq t) = \hat{u}(\mathbf{x}_t) \tag{38.18}$$

depending only on the value of the function at the last observed time. Suppose the stochastic equation

$$d\mathbf{X}_t^{\hat{u}} = \mathbf{b}(\mathbf{X}_t^{\hat{u}}, \hat{u}(\mathbf{X}_t^{\hat{u}})) dt + \sigma(\mathbf{X}_t^{\hat{u}}, \hat{u}(\mathbf{X}_t^{\hat{u}})) d\mathbf{W}_t \tag{38.19}$$

has a solution for every initial value $\mathbf{X}_0^{\hat{u}} = \mathbf{x} \in G$, and $E(\tau_G^{\mathbf{x}, \hat{u}}) < \infty$.

Then the control strategy (38.18) maximizes the expectation (38.7) among all control strategies $u(t; \mathbf{x}_s, 0 \leq s \leq t)$ for which $E(\tau_G^{\mathbf{x}, u}) < \infty$ and the functional $g(\mathbf{x}_t, u(t; \mathbf{x}_s, 0 \leq s \leq t))$ is bounded.

Proof. We have to prove that for an arbitrary control strategy $u(t; \mathbf{x}_s, 0 \leq s \leq t)$ and $\mathbf{x} \in G$ we have:

$$E(\text{Gain}^{\mathbf{x}, \hat{u}}) \geq E(\text{Gain}^{\mathbf{x}, u}) \tag{39.1}$$

The first thing we are going to check is that the first expectation is equal to $v(\mathbf{x})$.

To do this, we consider the martingale

$$\hat{Y}_t = v(\mathbf{X}_t^{\mathbf{x}, \hat{u}}) - \int_0^t L_{\hat{u}(\mathbf{X}_s^{\mathbf{x}, \hat{u}})} v(\mathbf{X}_s^{\mathbf{x}, \hat{u}}) ds \tag{39.2}$$

(that this is a martingale follows from Itô's formula: it is Y_0 plus the stochastic integral in this formula). Note that, since the control strategy (38.18) depends on the value of the function \mathbf{x}_s , $0 \leq s \leq t$, at the last observed time t only (and does not depend on the values of \mathbf{x}_s for $s < t$), $\mathbf{X}_t^{\mathbf{x}, \hat{u}}$ is just an ordinary diffusion process with coefficients $\mathbf{b}(\mathbf{x}, \hat{u}(\mathbf{x}))$ and $\sigma(\mathbf{x}, \hat{u}(\mathbf{x}))$. So, considering this martingale first at the bounded stopping time $\min(\tau_G^{\mathbf{x}, \hat{u}}, t_*)$, and then, by limit passage as $t_* \rightarrow \infty$, at $\tau_G^{\mathbf{x}, \hat{u}}$, we get, the usual way,

$$E\left(\varphi(\mathbf{X}_{\tau_G^{\mathbf{x}, \hat{u}}}^{\mathbf{x}, \hat{u}}) - \int_0^{\tau_G^{\mathbf{x}, \hat{u}}} L_{\hat{u}(\mathbf{X}_t^{\mathbf{x}, \hat{u}})} v(\mathbf{X}_t^{\mathbf{x}, \hat{u}}) dt\right) = v(\mathbf{x}). \tag{39.3}$$

By (38.17),

$$L_{\hat{u}(\mathbf{x})}v(\mathbf{x}) = \mathfrak{L}u(\mathbf{x}) - g(\mathbf{x}, \hat{u}(\mathbf{x})) = 0 - g(\mathbf{x}, \hat{u}(\mathbf{x})) = -g(\mathbf{x}, \hat{u}(\mathbf{x})); \quad (39.4)$$

and so

$$E\left(\varphi(\mathbf{X}_{\tau_G^{\mathbf{x}, \hat{u}}}) + \int_0^{\tau_G^{\mathbf{x}, \hat{u}}} g(\mathbf{X}_t^{\mathbf{x}, \hat{u}}, \hat{u}(\mathbf{X}_t^{\mathbf{x}, \hat{u}})) dt\right) = v(\mathbf{x}). \quad (39.5)$$

Now let us go to the right-hand side of (39.1). Consider the martingale

$$Y_t = v(\mathbf{X}_t^{\mathbf{x}, u}) - \int_0^t L_{u(s; \mathbf{X}_p^{\mathbf{x}, u}, 0 \leq p \leq s)}v(\mathbf{X}_s^{\mathbf{x}, u}, u(s; \mathbf{X}_p^{\mathbf{x}, u}, 0 \leq p \leq s)) ds. \quad (39.6)$$

Here the controlled process $\mathbf{X}_t^{\mathbf{x}, u}$ is not, generally, a diffusion process; but its coefficients depend only on the past, which is all we need to establish that a stochastic integral is a martingale. Applying the same device, we get:

$$v(\mathbf{x}) = E\left(\varphi(\mathbf{X}_{\tau_G^{\mathbf{x}, u}}) - \int_0^{\tau_G^{\mathbf{x}, u}} L_{u(t; \mathbf{X}_s^{\mathbf{x}, u}, 0 \leq s \leq t)}v(\mathbf{X}_t^{\mathbf{x}, u}, u(t; \mathbf{X}_s^{\mathbf{x}, u}, 0 \leq s \leq t)) dt\right). \quad (39.7)$$

For every $u \in U$,

$$L_u v(\mathbf{x}, u) \leq \mathfrak{L}v(\mathbf{x}) - g(\mathbf{x}, u) = -g(\mathbf{x}, u), \quad (39.8)$$

so we have:

$$v(\mathbf{x}) \geq E\left(\varphi(\mathbf{X}_{\tau_G^{\mathbf{x}, u}}) + \int_0^{\tau_G^{\mathbf{x}, u}} g(\mathbf{X}_t^{\mathbf{x}, u}, u(t; \mathbf{X}_s^{\mathbf{x}, u}, 0 \leq s \leq t)) dt\right); \quad (39.9)$$

which was to be proved.

By the way, it follows from

$$v(\mathbf{x}) = \max_{\text{all strategies } u} E(\text{Gain}^{\mathbf{x}, u}) \quad (39.10)$$

that the solution of the nonlinear problem (38.17) is unique – at least within the class of functions described in the Theorem.

We can modify Theorem 38.1 so as to get rid of the requirement of $v(\mathbf{x})$ being extendable smoothly to the whole \mathbb{R}^r as we did in Microtheorem 32.2; but anyway in our examples the extension will be very easy, so I don't either prove or even formulate this modification of our Theorem.

Theorem 39.1. *Let us consider, instead of the nonlinear differential operator \mathfrak{L} , the operator*

$$\mathfrak{L}'f(\mathbf{x}) = \min_{u \in U} (L_u f(\mathbf{x}) + g(\mathbf{x}, u)) \quad (39.11)$$

and the boundary-value problem $\mathfrak{L}'v(\mathbf{x}) = 0$, $\mathbf{x} \in G$, $v(\mathbf{x}) = \varphi(\mathbf{x})$, $\mathbf{x} \in \partial G$. For $\mathbf{x} \in G$, let $\hat{u}(\mathbf{x})$ be the value of the control parameter u at which the minimum $\min_{u \in U} (L_u v(\mathbf{x}) + g(\mathbf{x}, u))$ is reached.

Under the same conditions as in Theorem 38.1, the (Markov) control strategy $\hat{u}(t; \mathbf{x}_s$, $0 \leq s \leq t) = \hat{u}(\mathbf{x}_t)$ minimizes the expectation

$$E\left(\varphi(\mathbf{X}_{\tau_G^{\mathbf{x}, u}}) + \int_0^{\tau_G^{\mathbf{x}, u}} g(\mathbf{X}_t^{\mathbf{x}, u}, u(t; \mathbf{X}_s^{\mathbf{x}, u}, 0 \leq s \leq t)) dt\right). \quad (39.12)$$

The proof is the same as that of Theorem 38.1.