Lecture note 40. Controlled diffusion processes. Examples.

Let us consider some examples.

Example 40.1. Let the dimension $r = 1$, $G = (a, c)$, $U = (-\infty, \infty)$; the diffusion coefficient

$$a(x, u) \equiv 2 \qquad (\text{i. e., } \sigma(x, u) \equiv \sqrt{2}), \tag{40.1}$$

and the drift

$$b(x, u) = 2u. \tag{40.2}$$

Suppose we want to leave the interval (a, c) in the shortest possible time on the average, so our loss per time unit with the control equal to 0 is equal to 1. Using control $u \neq 0$ may shorten the time spent in the interval; but we have to pay for using the control at the rate of u^2 per unit of time. So our total loss per unit of time is

$$g(x, u) = g(u) = 1 + u^2. \tag{40.3}$$

We gain nothing at the time $\tau_{(a,\,c)}$ of leaving the interval, that is,

$$\varphi(a) = \varphi(c) = 0. \tag{40.4}$$

So we want to *minimize* the expectation

$$E\Big(\int_0^{\tau_{(a,\,c)}^{x,\,u}} g\big(u(t;\, X_s^{x,\,u},\, 0 \le s \le t)\big)\, dt\Big). \tag{40.5}$$

It stands to reason that we should drive our controlled process to the right if we are in the right half $\big((a + c)/2,\, c\big)$ of our interval, and to the left in its left half; but at what speed (what drift should we apply)? It seems also that we shouldn't apply big values of control variable near the middle of the interval (too expensive, and it is very uncertain which end of the interval our process will turn to).

According to Theorem 39.1 (the version for minimizing), we should consider the nonlinear operator

$$\mathfrak{L}' f(x) = \min_{-\infty < u < \infty} \big(f''(x) - 2u f'(x) + 1 + u^2\big). \tag{40.6}$$

Differentiating, we get that the minimum is reached at $u = -f'(x)$, and

$$\mathfrak{L}' f(x) = f''(x) - f'(x)^2 + 1. \tag{40.7}$$

Let us solve the equation

$$\mathfrak{L}' v(x) = v''(x) - v'(x)^2 + 1 = 0, \tag{40.8}$$

i. e., $v''(x) = v'(x)^2 - 1$.

Introducing the new unknown function $w(x) = v'(x)$, we can write:

$$w'(x) = w(x)^2 - 1, \qquad \frac{dw}{w^2-1} = dx, \qquad \frac{1}{2}\ln\left|\frac{w-1}{w+1}\right| = x - C, \qquad \frac{w-1}{w+1} = \pm e^{2(x-C)}, \tag{40.9}$$

$$w(x) = \frac{1+e^{2(x-C)}}{1-e^{2(x-C)}} \tag{40.10}$$

or

$$w(x) = \frac{1-e^{2(x-C)}}{1+e^{2(x-C)}}. \tag{40.11}$$

If $w(x) = v'(x)$ is given by formula (40.10) (of course, the constant C couldn't be in the interval (a, c), because $w(x)$ wouldn't make any sense for $x = C$), the sign of $v'(x)$ is either positive everywhere in the interval (a, c), or everywhere negative. Neither of these possibilities is compatible with the boundary conditions $v(a) = v(c) = 0$. So it is formula (40.11):

$$v'(x) = \frac{1-e^{2(x-C)}}{1+e^{2(x-C)}} = \frac{e^{-(x-C)} - e^{x-C}}{e^{-(x-C)} + e^{x-C}} = \frac{-\sinh(x-C)}{\cosh(x-C)} = -\tanh(x-C) \tag{40.12}$$

(draw the graph of the hyperbolic tangent),

$$v(x) = \int_a^x \big(-\tanh(y-C)\big)\,dy = -\int_a^x \frac{e^{y-C} - e^{-(y-C)}}{e^{y-C} + e^{-(y-C)}}\,dy \tag{40.13}$$

(because $v(a) = 0$). We have:

$$v(c) = -\int_a^c \tanh(x-C)\,dx = 0. \tag{40.14}$$

This can only be if C is the middle of the interval: $C = (a+c)/2$. So

$$v(x) = -\int_a^x \tanh\big(y - \frac{a+c}{2}\big)\,dy \tag{40.15}$$

(sorry, couldn't "take" this integral; i. e., express it through elementary functions). The optimal control $\hat{u}(x)$ is the point u at which the minimum $\min_u\big(v''(x) + 2uv'(x) + 1 + u^2\big)$ is reached, i. e.,

$$\hat{u}(x) = -v'(x) = \tanh\big(x - \frac{a+c}{2}\big) = \frac{e^{x-(a+c)/2} - e^{(a+c)/2-x}}{e^{x-(a+c)/2} + e^{(a+c)/2-x}} \tag{40.16}$$

(draw the graph of this function).

So indeed, the optimal control is equal to 0 at the middle of the interval; positive in its right half (we are driving the process to the right so it reaches the right end sooner: there is more chance with the right end that with the left end), and negative in $\big(a, \dfrac{a+c}{2}\big)$; nowhere does the optimal control reach the values $u = \pm 1$ or more in absolute value.

Example 40.2. Again $r = 1$, $G = (0, c)$, and $U = \{0, 1\}$. The diffusion coefficient

$$a(x, u) \equiv 1, \tag{40.17}$$

and the drift

$$b(x, u) = u \tag{40.18}$$

(that is, for stretches of time when we are using the control $u = 0$, our controlled process is just the standard Wiener process, and for those with $u = 1$ it is the Wiener process plus the uniform motion with velocity 1). We want to *maximize* the time spent by our process in the interval before the first exit time, minus the cost of control, which is at the rate $1/2$ per time unit for the control $u = 1$, and 0 for control u equal to 0. That is, we want to maximize the expectation

$$E\Big(\int_0^{\tau_{(0,\,c)}^{x,\,u}} \big(1 - \tfrac{1}{2}\, u(t;\, X_s^{x,\,u},\, 0 \le s \le t)\big)\; dt\Big). \tag{40.19}$$

We consider the nonlinear differential operator

$$\begin{aligned} \mathfrak{L} f(x) &= \min\big(\frac{1}{2}\, f''(x) + 1,\; \frac{1}{2}\, f''(x) + f'(x) + \frac{1}{2}\big) \\ &= \begin{cases} \ \tfrac{1}{2}\, f''(x) + 1, & f'(x) \le 1/2, \\ \tfrac{1}{2}\, f''(x) + f'(x) + 1/2, & f'(x) \ge 1/2. \end{cases} \end{aligned} \tag{40.20}$$

We determine the optimal control by solving the boundary-value problem $\mathfrak{L} v(x) = 0$, $0 < x < c$, $v(0) = v(c) = 0$.

It seems plausible that the optimal control should be $u = 1$ in some left part $(0, d)$ of our interval $(0, c)$ (we should try to drive the process away from the end 0 that it is about to reach), and $u = 0$ for the points to the right of d: we shouldn't drive the process to the right if it is already pretty far to the right. These two parts should be characterized by $v'(x) \ge 1/2$ for $x \le d$, and $v'(x) \le 1/2$ for $x \ge d$ (at the point d itself it should be $v'(d) = 1/2$).

If we are not satisfied by this vague reasoning, or rather these vague feelings, but succeed in finding a solution of our boundary-value problem starting from them, it is OK: Theorem 38.1 states that whatever unlawful way we used to find our solution, still the problem is solved.

So we start looking for a function $v(x)$, $0 \le x \le c$, that satisfies the equation

$$\tfrac{1}{2}\, v''(x) + v'(x) + 1/2 = 0 \tag{40.21}$$

in the left part of our interval, say, from $x = 0$ to x equal to some $d > 0$, and $v'(x) \ge 1/2$ in this part of the interval; and this function satisfies the equation

$$\tfrac{1}{2}\, v''(x) + 1 = 0 \tag{40.22}$$

in the interval from d to c, with $v'(x) \leq 1/2$ in this part of our interval. In addition, $v(x)$ has to satisfy the boundary conditions

$$v(0) = v(c) = 0 \tag{40.23}$$

and what is reasonable to call the *gluing conditions* at the point d: the left-hand limits and the right-hand ones of the function and its derivatives have to coincide:

$$v(d^-) = v(d^+), \qquad v'(d^-) = v'(d^+), \tag{40.24}$$

so that a solution of the equation (40.21) and one of equation (40.22) are *glued together* to form a smooth function.

Why didn't I include here the equality $v''(d^-) = v''(d^+)$? We do require that our function $v(x)$ should have a continuous *second* derivative too.

This is because, since in the interval $(0, d)$ the derivative $v'(x) \geq 1/2$, and in (d, c) it is $\leq 1/2$, we have that the common value

$$v'(d) = v'(d^-) = v'(d^+) = 1/2; \tag{40.25}$$

and from equations (40.21), (40.22) we get:

$$v''(d^-) = -2v'(d^-) - 1 = -2 = v''(d^+): \tag{40.26}$$

this is satisfied automatically provided the second equality in (40.24) holds.

So we have five numbers to determine: the dividing point d, two constants determining the solution of (40.21), and two for the solution of (40.22). And we have five equations for them: two boundary conditions (40.23), and three more:

$$v(d^-) = v(d^+), \qquad v'(d^-) = v'(d^+) = 1/2. \tag{40.27}$$

Of course, not every system of five equations with five unknowns has a solution (we needn't bother about the solution being *unique*: automatically, a solution of the boundary-value problem for the equation $\mathfrak{L}v(x) = 0$ *is* unique). Even not every *linear* system of five equations with five unknowns has a solution; and our system is *not* linear: *the unknown* d enters it through a non-linear function v.

But we can try to exclude one or two unknowns.