Lecture note 41. Examples, continued.

We are considering Example 40.2.

We are looking for a function $v(x)$, $0 \le x \le c$, that satisfies the equation

$$\tfrac{1}{2}\, v''(x) + v'(x) + 1/2 = 0 \tag{40.21}$$

in the left part of our interval, from $x = 0$ to x equal to some $d > 0$, and $v'(x) \ge 1/2$ in this part of the interval; and satisfying the equation

$$\tfrac{1}{2}\, v''(x) + 1 = 0 \tag{40.22}$$

in the interval from d to c, with $v'(x) \le 1/2$ in this part of our interval. In addition, $v(x)$ has to satisfy the boundary conditions

$$v(0) = v(c) = 0 \tag{40.23}$$

and the gluing conditions at the point d: the left-hand limits and the right-hand ones of the function and its derivatives have to coincide:

$$v(d^-) = v(d^+), \qquad v'(d^-) = v'(d^+) = 1/2. \tag{40.24}$$

So we have five numbers to determine: the dividing point d, two constants determining the solution of (40.21), and two for the solution of (40.22). And we have five equations for them: two boundary conditions (40.23), and three gluing conditions (40.24).

Let us try to exclude one or two unknowns.

For $0 \le x < d$ we have:

$$v(x) = -\frac{x}{2} + C_1 + C_2 e^{-2x}, \tag{41.1}$$

where C_1 and C_2 are arbitrary constants; and in the right part of the interval, from the point d to its right end,

$$v(x) = -x^2 + C_3 + C_4 x. \tag{41.2}$$

Using the boundary conditions (40.23), we get that $C_1 = -C_2$, $C_3 = c^2 - C_4 c$. The condition

$$v'(d^-) = -1/2 - 2C_2 e^{-2d} = 1/2 \tag{41.3}$$

yields $C_2 = -e^{2d}/2$; the condition

$$v'(d^+) = -2d - C_4 = 1/2, \tag{41.4}$$

the value of $C_4 = -2d - 1/2$. So all unknowns are expressed in terms of the unknown d.

The first equality in (40.24), expressed in terms of d, is

$$-d/2 - 1/2 + e^{2d}/2 = (c-d)^2 - (c+d)/2, \tag{41.5}$$

$$e^{2d} - 2(c-d)^2 + c - 1 = 0. \tag{41.6}$$

The function of d in the left-hand side increases from the value $-2c^2 + c$ at $d = 0$ to $e^{2c} + c - 1 > 0$ at $d = c$. So a solution d between 0 and c (and a unique one) exists if and only if $-2c^2 + c \le 0$, $c \ge 1/2$.

What about $c < 1/2$?? In this case no solution given by two formulas (41.1), (41.21) in two parts of the interval $(0, c)$ exists. But in this case the function

$$v(x) = (c-x)x, \tag{41.7}$$

being the solution of the boundary-value problem

$$\begin{aligned} &\tfrac{1}{2}\, v'' + 1 = 0, \\ &v(0) = v(c) = 0, \end{aligned} \tag{41.8}$$

solves at the same time the boundary-value problem $\mathfrak{L}v(x) = 0$, $v(0) = v(c) = 0$, because the derivative $v'(x) < 1/2$ everywhere in the interval $(0, c)$.

So for $c \le 1/2$ the maximum gain function $v(x)$ is given by formula (41.7), and the optimal control is $u \equiv 0$: no expensive drift should be added since our process will anyway leave the small interval $(0, c)$ very soon.

For $c > 1/2$ we have to solve the nonlinear equation (41.33). This can be done, e. g., by Newton's method. So, for $c = 10$ I took a calculator and got $d = 2.0032$: the optimal control is driving the process to the right adding 1 to its drift to the left of the point 2.0032, and leaving it as a pure Wiener process to the right of this point. The maximum expected time of the pure Wiener process staying in the interval $(0, 10)$ is reached at the initial point $x = 5$, and it is equal to $m(5) = (10-5) \cdot 5 = 25$. The maximum gain for our *controlled* process with the same initial point is

$$v(5) = -5^2 + 100 + (2d + 1/2) \cdot (5 - 10) = 57.468. \tag{41.36}$$

Some gain, compared to the identical-zero control.

Equations that are given by one formula in some unknown subregion D_1 of the region G considered, and by another one in some other subregion D_2 have been considered and applied, e. g., to problems about heat transfer in media consisting of two phases of a substance: solid and liquid, with melting or congealing occurring on the dividing surface ∂ between D_1 and D_2. On this surface (curve in the case of dimension 2) some gluing conditions have to be satisfied (but they will be of quite a different character from ours: e. g., they probably should include the requirement that the limits *of the unknown function*, not its derivatives, on both sides of ∂ are equal to a certain constant: the melting temperature). Such problems are considered in the theory of partial differential equations both for elliptic equations (modelling stationary temperature distributions) and for parabolic ones (considering the motion of the dividing surface ∂; such problems are more complicated, but much more interesting). We were able to find the solution in Example 40.2 because the position of the dividing "surface" – the dividing *point* in the one-dimensional case – was characterised by a one-dimensional variable; in the multidimensional case this unknown is *infinite*-dimensional. However, there *are* methods for finding solutions of such problems (of course, since we are not in the eighteenth or the nineteenth century, we are mainly interested in *numerical* methods of finding *approximate* solutions). Here the theory of partial differential equations and the theory of optimal control should be helping one another. For a given problem with several equations and gluing conditions at the dividing surfaces, we should look for an optimization problem for which the Bellman equation reduces to these equations plus the gluing conditions; if this is done, and we have a method that allows us to *improve* a strategy that is not optimal, this can be translated into a method of successive approximations for finding, approximately, the solution of the PDE problem. But of course, all this is complicated both from the differential-equations point of view and from the angle of stochastic optimal control.