Lecture note 42. Problems of optimal stopping.

Another class of optimal control problems, different from problems of the form (38.7), (38.6), are *problems of optimal stopping*.

Suppose we have a diffusion process $\boldsymbol{X}_t^{\boldsymbol{x}}$ with coefficients $\boldsymbol{b}(\boldsymbol{x})$, $\sigma(\boldsymbol{x})$, and the corresponding linear operator $Lf(\boldsymbol{x})$. We watch this process up to the time $\tau_G^{\boldsymbol{x}}$ of its leaving a region G; and the only way to control it is the choice, at every time, whether to stop the process, or watch it for some more time. That is, our control strategy is the choice of a stopping time $\tau \le \tau_G^{\boldsymbol{x}}$ at which we stop our process (we cannot stop the process at a time that is not a stopping time based on the current observations of the process, because at some time t we couldn't decide whether we should have stopped the process by this time, or not).

Suppose a continuous function $f(\boldsymbol{x})$ is given in the region together with its boundary: $\boldsymbol{x} \in G \cup \partial G$; and on stopping the process at the time τ we get paid the sum

$$\text{Gain}^{\boldsymbol{x}}(\tau) = f(\boldsymbol{X}_\tau^{\boldsymbol{x}}). \tag{42.1}$$

How to choose the stopping time $\tau \le \tau_G^{\boldsymbol{x}}$ that maximizes our expected gain:

$$E\big(\text{Gain}^{\boldsymbol{x}}(\tau)\big) = \max? \tag{42.2}$$

Theorem 42.1. *Suppose $\tau_G^{\boldsymbol{x}}$ is almost surely finite. Suppose a function $v(\boldsymbol{x})$, twice continuously differentiable in G, satisfies the following equalities and inequalities*:

$$\begin{aligned} v(\boldsymbol{x}) &\ge f(\boldsymbol{x}), && \boldsymbol{x} \in G, \\ v(\boldsymbol{x}) &= f(\boldsymbol{x}), && \boldsymbol{x} \in \partial G, \\ Lv(\boldsymbol{x}) &= 0 && \textit{for } \boldsymbol{x} \in G \textit{ such that } v(\boldsymbol{x}) > f(\boldsymbol{x}), \\ Lv(\boldsymbol{x}) &\le 0 && \textit{for } \boldsymbol{x} \in G \textit{ such that } v(\boldsymbol{x}) = f(\boldsymbol{x}); \end{aligned} \tag{42.3}$$

suppose the function $v(\boldsymbol{x})$ can be extended (as we usually suppose) *smoothly to the whole space $\mathbb{R}^r$.*

Then the optimal control consists in choosing τ as the first time

$$\hat{\tau}^{\boldsymbol{x}} = \min\{t \ge 0\colon\ \boldsymbol{X}_t^{\boldsymbol{x}} \in A\} \tag{42.4}$$

at which the process reaches the closed set

$$A = \{\boldsymbol{x} \in G \cup \partial G\colon\ v(\boldsymbol{x}) = f(\boldsymbol{x})\}; \tag{42.5}$$

and $v(\boldsymbol{x})$ is our expected gain under the optimal control (optimal stopping):

$$v(\boldsymbol{x}) = E\big(\text{Gain}^{\boldsymbol{x}}(\hat{\tau}^{\boldsymbol{x}})\big) = \max_{\text{all stopping times } \tau \le \tau_G^{\boldsymbol{x}}} E\big(\text{Gain}^{\boldsymbol{x}}(\tau)\big). \tag{42.6}$$

By the way, it follows from this theorem that the solution of the problem (42.3) is unique.

Before giving the proof of the theorem, let me start with an example:

Example 42.1. Let $r = 1$, $G = (a, b)$, and the diffusion process considered the standard one-dimensional Wiener process; the operator L is given by $Lf(x) = \frac{1}{2} f''(x)$. Given a function $f(x)$, $a \le x \le b$, we take a thread, put it over the graph of the function f, and pull down at the ends of the thread; when it is stretched, it will show us the function $v(x)$: clearly everywhere $v(x) \ge f(x)$; the set A will consist of points x at which the equality takes place. Over the points x outside the set A the thread is stretched to be a straight line, and its second derivative is 0. Over the points $x \in A$ the graph of the function $f(x)$ (described by the thread) must be concave downwards: otherwise the thread would be pulled up.

So next time we'll have the proof of our theorem (it is pretty simple); a concrete example; and we'll see that something is not very good.