Lecture note 43. Problems of optimal stopping, continued.

Proof of Theorem 42.1. First of all, because of the second of the formulas (42.3) we have that $A \supseteq \partial G$; so $\hat{\tau}^{\boldsymbol{x}} \le \tau_G^{\boldsymbol{x}}$. Let us prove the first equality in (42.6) first.

As usual, we consider the martingale

$$Y_t = v(\boldsymbol{X}_t^{\boldsymbol{x}}) - \int_0^t Lv(\boldsymbol{X}_s^{\boldsymbol{x}})\, ds, \tag{43.1}$$

its value at the bounded stopping time $\min(\hat{\tau}_G^{\boldsymbol{x}}, t_*)$, and, by limit passage as $t_* \to \infty$, we find that

$$E(Y_{\hat{\tau}^{\boldsymbol{x}}}) = E\Big(v(\boldsymbol{X}_{\hat{\tau}^{\boldsymbol{x}}}^{\boldsymbol{x}}) - \int_0^{\hat{\tau}^{\boldsymbol{x}}} Lv(\boldsymbol{X}_t^{\boldsymbol{x}})\, dt\Big) = E(Y_0) = v(\boldsymbol{x}). \tag{43.2}$$

The formulas (42.3) can be rewritten as

$$\begin{aligned} v(\boldsymbol{x}) &> f(\boldsymbol{x}), \qquad \boldsymbol{x} \in G \setminus A,\\ v(\boldsymbol{x}) &= f(\boldsymbol{x}), \qquad \boldsymbol{x} \in A,\\ Lv(\boldsymbol{x}) &\le 0 \qquad \text{for } \boldsymbol{x} \in G,\\ Lv(\boldsymbol{x}) &= 0 \qquad \text{for } \boldsymbol{x} \in G \setminus A. \end{aligned} \tag{43.3}$$

Since $\boldsymbol{X}_{\hat{\tau}^{\boldsymbol{x}}}^{\boldsymbol{x}} \in A$, we have $v(\boldsymbol{X}_{\hat{\tau}^{\boldsymbol{x}}}^{\boldsymbol{x}}) = f(\boldsymbol{X}_{\hat{\tau}^{\boldsymbol{x}}}^{\boldsymbol{x}})$. And $\boldsymbol{X}_t^{\boldsymbol{x}} \in G \setminus A$ for $0 \le t < \hat{\tau}^{\boldsymbol{x}}$, so $Lv(\boldsymbol{X}_t^{\boldsymbol{x}}) = 0$ for these t. So we have:

$$v(\boldsymbol{x}) = E\Big(v(\boldsymbol{X}_{\hat{\tau}^{\boldsymbol{x}}}^{\boldsymbol{x}}) - \int_0^{\hat{\tau}^{\boldsymbol{x}}} Lv(\boldsymbol{X}_t^{\boldsymbol{x}})\, dt\Big) = E\big(f(\boldsymbol{X}_{\hat{\tau}^{\boldsymbol{x}}}^{\boldsymbol{x}})\big). \tag{43.4}$$

Now we have to prove that for every stopping time $\tau \le \tau_G^{\boldsymbol{x}}$

$$E\big(f(\boldsymbol{X}_\tau^{\boldsymbol{x}})\big) \le v(\boldsymbol{x}). \tag{43.5}$$

We use the same martingale (43.1), but a different stopping time, getting

$$v(\boldsymbol{x}) = E\Big(v(\boldsymbol{X}_\tau^{\boldsymbol{x}}) - \int_0^\tau Lv(\boldsymbol{X}_t^{\boldsymbol{x}})\, dt\Big). \tag{43.6}$$

Using the first three lines in (43.3), we get (43.5).

The theorem is proved.

We can consider modifications of this problem, e. g., discounting our gain with the lapse of time:

$$\text{Gain}^{\boldsymbol{x}}(\tau) = e^{-\beta\tau} f(\boldsymbol{X}_\tau^{\boldsymbol{x}}); \tag{43.7}$$

or combining control by means of stopping with the control as we had in Lecture notes 38 – 41, with the gain functional of the form

$$\text{Gain}^{\boldsymbol{x},\,u}(\tau) = f(\boldsymbol{X}_\tau^{\boldsymbol{x},\,u}) + \int_0^\tau g\big(\boldsymbol{X}_t^{\boldsymbol{x},\,u},\ u(t;\ \boldsymbol{X}_s^{\boldsymbol{x},\,u},\ 0 \le s \le t)\big)\ dt; \tag{43.8}$$

but better we return to Example 42.1 that we started considering.

Let us consider a concrete example: $(a, b) = (0, 2\pi)$, and the function $f(x) = e^x \sin x$ (draw its graph as realistically as you can). We have:

$$f'(x) = e^x(\sin x + \cos x), \tag{43.9}$$

$$f''(x) = 2e^x \cos x. \tag{43.10}$$

Near the ends of the interval the second derivative is positive, the graph is concave up, and the thread (the function $v(x)$) cannot follow the function f. So there will be two stretches with our thread stretched to a straight line: from 0 to some c, and from some $d > c$ to 2π (show this in your picture). Between the points c and d the thread will follow the graph of the function f: $v(x) = f(x)$, the set A being $\{0\} \cup [c, d] \cup \{2\pi\}$. At the points c and d the slope of the segments of straight line ending at these points will be equal to $f'(c)$, $f'(d)$: otherwise the function $v(x)$ would not be differentiable at these points (even once). So we have:

$$v(x) = m_1 \cdot x,\ \ 0 \le x \le c, \qquad v(c) = m_1 \cdot c = f(c), \ \ v'(c) = m_1 = f'(c), \qquad f'(c) \cdot c = f(c), \tag{43.11}$$

$$c \cdot (\sin c + \cos c) = \sin c, \tag{43.12}$$

$$v(x) = m_2 \cdot (x - 2\pi),\ \ d \le x \le 2\pi, \tag{43.13}$$

$$(d - 2\pi) \cdot (\sin d + \cos d) = \sin d. \tag{43.14}$$

Taking a calculator and solving the equations (43.12), (43.14), say, using Newton's method, we get:

$$c \approx 2.0428, \qquad d \approx 2.4716. \tag{43.15}$$

So the set A is given by

$$A = \{0\} \cup [2.0428, 2.4716] \cup \{2\pi\}, \tag{43.16}$$

and our thread describes the graph of the function

$$v(x) = \begin{cases} 3.3625\,x, & 0 \le x \le 2.0428, \\ e^x \sin x, & 2.0428 \le x \le 2.4716, \\ 1.9292(2\pi - x), & 2.4716 \le x \le 2\pi. \end{cases} \tag{43.17}$$

It would seem that it follows from Theorem 42.1 that the optimal stopping is stopping at reaching the set A: either one of the ends 0, 2π, or any of the points of the interval

$[2.0428, 2.4716]$ (if we start from one of the points of this interval, we should start instantly; and if we start outside, we stop at reaching one of the points 0, 2π, 2.0428, or 2.4716).

However, Theorem 42.1 *cannot be applied* to our function $v(x)$: this function is continuous and once continuously differentiable all right, but it is not *twice* continuously differentiable: its left-hand second derivative is 0 at the point $c = 2.0428$, but its right-hand second derivative is equal to

$$2e^{2.0428} \cdot \cos 2.0428 = -7.0130 \neq 0, \tag{43.18}$$

and the left-hand second derivative at the point $d = 2.4716$ is equal to

$$2e^{2.4716} \cdot \cos 2.4716 = -18.5632. \tag{43.19}$$

We seem to have no hope to find a twice continuously differentiable function solving our problem (the thread reasoning seems to be too convincing); so better we try to modify Theorem 42.1 to include some functions that are once, but not twice continuously differentiable. (After all, our function (43.17) violates this requirement at only two points, and what are just two points? almost nothing! However, it turns out that we cannot drop the requirement of the *first* derivative being continuous – even if it is at the measly two points.)

Our Theorem 42.1 rests on the theorem about Itô's formula; so it is this that we should try to modify. I'll formulate and prove the modified version of this theorem in the one-dimensional case, for a function depending on the spatial argument only.

Theorem 43.1. *Let X_t be a one-dimensional stochastic process having a stochastic differential*

$$dX_t = f(t,\omega)\,dt + g(t,\omega)\,dW_t. \tag{43.20}$$

Let $F(x)$ be a function that is once continuously differentiable, and let its second derivative exist except at finitely many points x_1, ..., x_m, at which one-sided limits $F''(x_i^-) = \lim_{x\to x_i^-} F''(x)$, $F''(x_i^+) = \lim_{x\to x_i^+} F''(x)$ exist. Suppose that the function $F(x)$ is equal to 0 outside some finite interval, and that for every finite interval $[A, B]$ there exists a constant such that the random functions $f(t,\omega)$, $g(t,\omega)$ remain bounded by this constant for t for which $X_t(\omega) \in [A, B]$ (this requirement is for our stochastic integral to make sense).

Then

$$dF(X_t) = F'(X_t)\cdot g(t,\omega)\,dW_t + \Big[\frac{1}{2}\,F''(X_t)\cdot g(t,\omega)^2 + F'(X_t)\cdot f(t,\omega)\Big]\,dt, \tag{43.21}$$

where F'' is replaced at the points x_i either with $F''(x_i^-)$, or with $F''(x_i^+)$, or with an arbitrary number.

Proof. Let $\varphi(x)$ be a continuously differentiable function on an interval $[l, r]$ (l and r for the *l*eft and the *r*ight ends of the interval). Let us take this interval so that its length is smaller than the smallest distance between different points x_i. Let us define a function $F_\varphi(x)$ by

$$F_\varphi(x) = \int_l^r \varphi(y)\cdot F(x+y)\;dy. \tag{43.22}$$

Clearly the function $F_\varphi(x)$ is continuous; let us differentiate it.

Certainly you know the following result (so I am not going to prove it):

Microtheorem 43.1. *Let* $f(x,y)$, $a < x < b$, $c \le y \le d$, *be a continuous function, and let the partial derivative* $\dfrac{\partial f}{\partial x}(x,y)$ *be also continuous in* $(a,b) \times [c,d]$.

Then for $a < x < b$

$$\frac{d}{dx}\int_c^d f(x,y)\ dy = \int_c^d \frac{\partial f}{\partial x}(x,y)\ dy. \tag{43.23}$$

In the lecture I also formulated a microtheorem about the derivative

$$\frac{d}{dx}\int_{c(x)}^{d(x)} f(x,y)\ dy; \tag{43.24}$$

but it seems to me now that better I do it in some different way.

We proceed to differentiate the function $F_\varphi(x)$:

$$F'_\varphi(x) = \int_l^r \frac{\partial f}{\partial x}\bigl(\varphi(y)\cdot F(x+y)\bigr)\ dy = \int_l^r \varphi(y)\cdot F'(x+y)\ dy. \tag{43.25}$$

So the first derivative $F'_\varphi(x)$ exists everywhere, and is continuous.

Unfortunately, we cannot apply Microtheorem 43.1 to finding the second derivative, because the function $\varphi(y)\cdot F'(x+y)$ may not have a continuous derivative in the rectangle $(a,b)\times[l,r]$. Nevertheless we can try to prove, without relying on this result, that

$$F''_\varphi(x) = \int_l^r \varphi(y)\cdot F''(x+y)\ dy. \tag{43.26}$$

The integrand here is not defined at the points $y = x_i - x$ (at most one of which can belong to the interval $[l,r]$); but it is continuous outside these points and has one-sided limits at this point, so the integral (in which we replace the non-existent $F''(x_i)$ by an arbitrary number) exists, and is a continuous function of x (and the integral does not depend on what we replaced $F''(x_i)$ with); the only thing is whether this function is indeed the derivative of $F'_\varphi(x)$ (whether the equality (43.26) holds).

And we postpone this till the next lecture.