Lecture note 45. Optimal stopping problems.

Let us consider an application to finance: stock options.

We considered in Lecture note 11 the diffusion process X_t modelling oscillations of price of some sort of stock: it was defined as the solution of the stochastic equation

$$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t \tag{45.1}$$

(see formula (11.10)); the solution starting from the point $x_0 > 0$ at time t_0 was represented also in the form

$$X_t = e^{Y_t}, \tag{45.2}$$

Y_t being the diffusion process satisfying the equation

$$dY_t = \mu_0\,dt + \sigma\,dW_t, \tag{45.3}$$

starting from the point $y_0 = \ln x_0$, where $\mu_0 = \mu - \sigma^2/2$ (see formula (11.21)).

The diffusion and drift coefficients for the process X_t are

$$a_X(t,x) = \sigma^2 \cdot x^2, \qquad b_X(t,x) = \mu \cdot x, \tag{45.4}$$

for the process Y_t

$$a_Y(t,y) \equiv \sigma^2, \qquad b_Y(t,y) \equiv \mu_0 \tag{45.5}$$

(so the process Y_t is much simpler: it is just the Wiener process multiplied by a constant and subject to a drift with a constant speed).

Let us consider a selling option with fixed price per stock unit D. Our gain from exercising the option at the time at which the market price of the stock in question is x is

$$G_{\mathrm{s}}(x) = \begin{cases} D - x, & x \le D, \\ 0, & x \ge D \end{cases} \tag{45.6}$$

(see formula (11.24)).

Suppose we are allowed to exercise our option at any time between the time t_0 at which we are acquiring it, and some fixed time $t_1 > t_0$. Of course, if we cannot observe the future prices of our stock, we can only exercise our option at a *stopping time.* So maximizing our expected gain from using our selling option is a typical optimal stopping problem (though not in the too-simple formulation of Lecture note 42): we consider a diffusion process X_t with (t, X_t) being in the two-dimensional region $[t_0,\, t_1] \times (0, \infty)$; and we want to find a stopping time $\tau \le t_1$ (t_1 being the time at which (t, X_t) leaves our region) maximizing the expected gain:

$$E\big(G_{\mathrm{s}}(X_\tau)\big) = \max. \tag{45.7}$$

We know, in principle, how to solve this maximization problem: we look for a solution $v(t,x)$, $t_0 \le t \le t_1$, $x \in (0,\infty)$, satisfying the conditions of Theorem 43.1$'$ (that is, once continuously differentiable in x everywhere, and once in t and twice in x with exception of several lines; see Lecture note 44) of the problem

$$\begin{gathered} v(t,x) \ge G_{\mathrm{s}}(x), \quad t_0 \le t \le t_1,\ 0 < x < \infty, \qquad v(t_1,x) = G_{\mathrm{s}}(x), \\ \frac{\partial v}{\partial t} + \frac{\sigma^2 x^2}{2}\,\frac{\partial^2 v}{\partial x^2}(t,x) + \mu x\,\frac{\partial v}{\partial x}(t,x) \le 0 \quad \text{except on the lines mentioned}, \\ \frac{\partial v}{\partial t} + \frac{\sigma^2 x^2}{2}\,\frac{\partial^2 v}{\partial x^2}(t,x) + \mu x\,\frac{\partial v}{\partial x}(t,x) = 0 \quad \text{for } (t,x) \text{ for which } v(t,x) > G_{\mathrm{s}}(x). \end{gathered} \tag{45.8}$$

The optimal stopping is that at the time

$$\hat{\tau} = \min\{t \le t_1 \colon\ (t, X_t) \in A\}, \tag{45.9}$$

where

$$A = \{(t,x) \in [t_0,\, t_1] \times (0,\infty) \colon\ v(t,x) = G_{\mathrm{s}}(x)\} \tag{45.10}$$

(so the set A necessarily contains the line $\{t_1\} \times (0,\infty)$).

This problem means finding a two-dimensional region $B \subseteq [t_0, t_1) \times (0,\infty)$ and a function $v(t,x)$ such that it satisfies the above-mentioned differentiability conditions, and

$$\begin{gathered} v(t,x) = G_{\mathrm{s}}(x), \quad (t,x) \in B \text{ or } t = t_1, \qquad v(t,x) \le G_{\mathrm{s}}(x), \quad (t,x) \notin B, \\ \frac{\partial v}{\partial t} + \frac{\sigma^2 x^2}{2}\,\frac{\partial^2 v}{\partial x^2}(t,x) + \mu x\,\frac{\partial v}{\partial x}(t,x) \le 0 \quad \text{for } (t,x) \in B, \\ \frac{\partial v}{\partial t} + \frac{\sigma^2 x^2}{2}\,\frac{\partial^2 v}{\partial x^2}(t,x) + \mu x\,\frac{\partial v}{\partial x}(t,x) = 0 \quad \text{outside } B \cup \partial B. \end{gathered} \tag{45.11}$$

In the particular case of the function $G_{\mathrm{s}}(x)$ given by formula (45.6), it stands to reason that the region B should have the form

$$B = \{(t,x) \colon\ 0 < x \le x(t)\}, \tag{45.12}$$

where $x(t)$ is some function: we sell our stock for the price of D per unit if the market price is low enough to make it profitable, and we refrain from exercising our option if the market price is higher than some threshold $x(t)$.

We know that the solution of even a linear problem: the Cauchy problem for a parabolic equation is not unique if we don't impose any restrictions on how fast the solution can grow at infinity (in our present case it should be rather as $x \to \infty$ and as $x \to 0^+$). The simplest restriction possible is to require that the solution $v(x)$ should be bounded. (This is one of the reasons for which I decided considering the *selling* option here rather than the buying option: the function $G_{\mathrm{b}}(x)$ given by formula (11.22) is unbounded, so we couldn't do with the simple restriction of boundedness of the solution $v(x)$.) Also we need some restrictions on the behavior as $x \to \infty$ and as $x \to 0^+$ of the derivatives: so that the stochastic integral makes sense; it is enough to require that $\dfrac{\partial v}{\partial x}(x) \cdot x$ be bounded.

Of course, the question arises of *how* to solve problem (45.11): how to find the function $x(t)$ of one variable and another one, $v(t, x)$, of two variables? We could speak about this – – not having enough time.

One thing we can say: it is easier to consider parabolic equations with constant coefficients than with variable ones; so it seems reasonable to go to the diffusion process Y_t through which the process X_t is expressed by means of formula (45.2). So for the process Y_t we have to maximize the expectation $E\big(g(Y_\tau)\big)$, where

$$g(y) = \begin{cases} D - e^y, & -\infty < y \le \ln D, \\ 0, & \ln D \le y < \infty \end{cases} \tag{45.13}$$

(make a picture of the graph of this function).