Lecture note 6. Stochastic integrals. Stochastic differentials. An example.

Quite obviously, stochastic integrals of random functions determined by the past (of the Wiener process) are linear with respect to their integrands: the equalities (5.1), (5.2) are satisfied. But in contrast with the case of step random functions, for which these formulas hold for *all* ω, in the general case they hold only *almost surely.* Since the stochastic integrals are defined by limit passage from integrals of step functions, these properties are proved just by this limit passage; and since mean-square limits are defined uniquely only almost everywhere, this is how the equalities hold. Also by limit passage from formulas (4.55), (4.56) for step random functions we get that for every mean-square continuous random function $f(t,\omega)$ determined by the past with $E\big(f(t,\omega)^2\big) < \infty$ we have:

$$E\Big(\int_a^b f(t,\omega)\; dW_t\Big) = 0, \tag{6.1}$$

$$E\Big(\Big(\int_a^b f(t,\omega)\; dW_t\Big)^2\Big) = E\Big(\int_a^b f(t,\omega)^2\, dt\Big) = \int_a^b E\big(f(t,\omega)^2\big)\; dt. \tag{6.2}$$

The second remark: We defined the stochastic integral $\int_a^b f(t,\omega)\; dW_t$ as the limit in mean of the integrals $\int_a^b f_{\mathfrak{T}}(t,\omega)\; dW_t$ of step random functions, $f_{\mathfrak{T}}(t,\omega)$ being defined by (5.28) (or (5.29)). This definition can, of course, be written in the form

$$\int_a^b f(t,\omega)\; dW_t = \underset{\max_{1\le t\le n}(t_i - t_{i-1})\to 0}{\text{l.i.m.}} \sum_{i=1}^n f(t_{i-1},\omega)\cdot(W_{t_i} - W_{t_{i-1}}). \tag{6.3}$$

That is, in the Riemann-Stieltjes sums we take the integrand at the left end t_{i-1} of every interval $[t_{i-1}, t_i]$ of the partition. Can any other point t_i^* be chosen in the interval from t_{i-1} to t_i to take the value $f(t_i^*,\omega)$ of the random function at it? No, in general, if we want our proof of mean-square convergence (of Theorem 5.1) to stand. Indeed, if we take, for example, in lieu of $f_{\mathfrak{T}}(t,\omega)$,

$$f^{\mathfrak{T}}(t,\omega) = f(t_1,\omega)\cdot I_{[t_0,\,t_1]}(t) + f(t_2,\omega)\cdot I_{(t_1,\,t_2]}(t) + ... + f(t_n,\omega)\cdot I_{(t_{n-1},\,t_n]}(t), \tag{6.4}$$

this step function won't be, in general, determined only by the past: its value, say, at the point $(t_i + t_{i-1})/2$ is equal to $f(t_i,\,\omega) = f(t_i;\, W_s,\, s \le t_i)$, which may depend on the values of W_s for s up to t_i rather than only up to $(t_i + t_{i-1})/2$.

But there are some classes of random functions $f(t,\omega)$ for which choosing arbitrary $t_i^* \in [t_{i-1}, t_i]$ is possible.

First of all, in the case of $f(t,\omega) = f(t)$, being, in fact, a non-random, deterministic function. The step function

$$f_{\mathfrak{T};\, t_1^*,\,...,\,t_n^*}(t) = \begin{cases} f(t_1^*), & t_0 \le t \le t_1, \\ f(t_2^*), & t_1 < t \le t_2, \\ \cdots\cdots\cdots\cdots & \\ f(t_n^*), & t_{n-1} < t \le t_n, \end{cases} \tag{6.5}$$

converges to $f(t)$ uniformly in $t \in [a, b]$ as $\max_{1 \le i \le n}(t_i - t_{i-1}) \to 0$, whatever way we choose $t_i^* \in [t_{i-1}, t_i]$; and the value of the function $f_{\mathfrak{T};\, t_1^*, \dots, t_n^*}(t)$ on the i-th interval *is* a functional of W_s, $s \le t_{i-1}$: namely, a constant one, *not* depending on the Wiener process (or whatever else) at all. So for non-random integrands we can take the sum $\sum_{i=1}^n f(t_i^*) \cdot (W_{t_i} - W_{t_{i-1}})$ instead of $\sum_{i=1}^n f(t_{i-1}) \cdot (W_{t_i} - W_{t_{i-1}})$.

A second class of integrands $f(t, \omega)$ for which we can guarantee that

$$\int_a^b f(t, \omega)\, dW_t = \operatorname*{l.i.m.}_{\max_{1 \le t \le n}(t_i - t_{i-1}) \to 0} \sum_{i=1}^n f(t_i^*, \omega) \cdot (W_{t_i} - W_{t_{i-1}}) : \tag{6.6}$$

the random functions (with finite expectation of the square, and mean-square continuous) that are determined by the past with some lag:

$$f(t, \omega) = f(t;\, W_s, s \le t'), \qquad \text{where } t' < t. \tag{6.7}$$

For example:

$$f(t, \omega) = W_{\max(t_0,\, t - 0.05)}. \tag{6.8}$$

Here, if $\max_{1 \le i \le n}(t_i - t_{i-1}) < 0.05$, we have $t_i^* - 0.05 < t_{i-1}$ for every i, and the random variable $f(t_i^*, \omega)$ *is* a functional of W_s, $s \le t_{i-1}$. So the limit (6.6) with all choices of $t_i^* \in [t_{i-1}, t_i]$ exists, and is independent on the choice of t_i^* (otherwise we would have that the limit (6.6) does not exist).

But for general random functions $f(t, \omega)$ determined by the past the definition (6.6) is impossible.

To be precise, *our method* of proving that the mean-square limit exists is not valid for the limit (6.6) for general integrands determined by the past.

Presently, we'll have some examples, and see whether the limit (6.6) exists in these examples.

My third remark. It irritates me when in a Calculus textbook the Riemann integral is defined only for continuous integrands. The Riemann integral can be defined for many discontinuous functions. For example, if $f(t)$, $a \le t \le b$, is continuous except at a point $c \in (a, b)$, and the one-sided finite limits $\lim_{t \to c^-} f(t)$, $\lim_{t \to c^+} f(t)$ exist (make a picture of a function that is continuous in $[a, b] \setminus \{c\}$, with different one-sided limits at c, and a value $f(c)$ at this point not equal to either of the limits), the Riemann integral still exists.

The same is the situation with stochastic integrals: *Let* $f(t, \omega)$, $a \le t \le b$, *be a random function determined by the past with* $E\big(f(t, \omega)^2\big) < \infty$. *Let this random function be mean-square continuous at all* $t \ne c$, *and let the one-sided mean-square limits*

$$\operatorname*{l.i.m.}_{t \to c^-} f(t, \omega) = F_1(\omega), \qquad \operatorname*{l.i.m.}_{t \to c^+} f(t, \omega) = F_2(\omega) \tag{6.9}$$

exist. Then the mean-square limit (6.3) *exists, and the stochastic integral* $\int_a^b f(t, \omega)\, dW_t$ *is defined.*

Proof. The random function $f_1(t,\omega)$, $a \le t \le c$, defined by

$$f_1(t,\omega) = \begin{cases} f(t,\omega), & a \le t < c, \\ F_1(\omega), & t = c, \end{cases} \tag{6.10}$$

is mean-square continuous on $[a,c]$. Therefore it is *uniformly* mean-square continuous, and for every positive ε there exists a positive δ_1 such that

$$s,t \in [a,c),\ |s-t| < \delta_1 \ \Rightarrow\ E\big(\big(f(t,\omega) - f(s,\omega)\big)^2\big) < \varepsilon. \tag{6.11}$$

Similarly, there exists a positive δ_2 such that

$$s,t \in (c,b],\ |s-t| < \delta_2 \ \Rightarrow\ E\big(\big(f(t,\omega) - f(s,\omega)\big)^2\big) < \varepsilon. \tag{6.12}$$

For two partitions $\mathfrak{T}$ with partition points t_i, $1 \le i \le n$, and $\mathfrak{T}'$ with partition points t'_i, $1 \le i \le n'$, and for a time point $t \in [a,b]$ let i be the number of the interval of the partition $\mathfrak{T}$ containing this point, and i' the same for the partition $\mathfrak{T}'$. We have:

$$f_{\mathfrak{T}}(t,\omega) - f_{\mathfrak{T}'}(t,\omega) = f(t_{i-1},\omega) - f(t'_{i'-1},\omega). \tag{6.13}$$

Suppose $\max_{1\le i\le n}(t_i - t_{i-1}) < \delta$, $\max_{1\le j\le n'}(t'_j - t'_{j-1}) < \delta$. The points t_{i-1}, $t'_{i'-1}$ in formula (6.13) are both to the left of the point t, at a distance less than δ from it, so $|t_{i-1} - t'_{i'-1}| < \delta$.

Let us take a positive $\delta \le \min(\delta_1, \delta_2)$.

For $a \le t < c$ we get from (6.11):

$$E\big(\big(f_{\mathfrak{T}}(t,\omega) - f_{\mathfrak{T}'}(t,\omega)\big)^2\big) < \varepsilon. \tag{6.14}$$

The same is true for $c + \delta \le t \le b$, by (6.12).

And for $t \in [c, c+\delta)$ we have:

$$\begin{aligned} &E\big(\big(f_{\mathfrak{T}}(t,\omega) - f_{\mathfrak{T}'}(t,\omega)\big)^2\big) \\ &\quad \le M = 4\max\big(\max_{a\le t\le c} E\big(f_1(t,\omega)^2\big),\ \max_{c\le t\le b} E\big(f_2(t,\omega)^2\big),\ E\big(f(c,\omega)^2\big)\big) \end{aligned} \tag{6.15}$$

(a finite maximum of the quadratic expectation exists for every mean-square continuous function on every finite closed interval). So we have for $\max_{1\le i\le n}(t_i - t_{i-1}) < \delta$, $\max_{1\le j\le n'}(t'_j - t'_{j-1}) < \delta$:

$$\begin{aligned} &E\Big(\big(\int_a^b f_{\mathfrak{T}}(t,\omega)\,dW_t - \int_a^b f_{\mathfrak{T}'}(t,\omega)\,dW_t\big)^2\Big) \\ &\qquad = \int_a^b E\big(\big(f_{\mathfrak{T}}(t,\omega) - f_{\mathfrak{T}'}(t,\omega)\big)^2\big)\,dt < \varepsilon\cdot(b-a) + M\cdot\delta. \end{aligned} \tag{6.16}$$

We chose δ so that it is $\le \min(\delta_1, \delta_2)$; but this seems to be not enough. Let us take positive δ still smaller: so that it is $\le \min(\delta_1, \delta_2, \varepsilon)$. Then we have:

$$E\Big(\big(\int_a^b f_{\mathfrak{T}}(t,\omega)\ dW_t - \int_a^b f_{\mathfrak{T}'}(t,\omega)\ dW_t\big)^2\Big) < \varepsilon \cdot (b - a + M), \tag{6.17}$$

and this is just as good as being less than ε.

So the stochastic integral exists for every random function determined by the past that has just one point of mean-square discontinuity with mean-square limits from the left and from the right at this point.

Of course, the same is true for a random function having any finite number of discontinuity points of this kind.

In particular, for the step random functions that we started with – only for step functions it is much simpler, and none of the above calculations are needed.

By the way, it follows from our considerations that the value of the integrand at one time point, or at finitely many points, does not affect the value of the stochastic integral (almost surely).

If W_t, $t \ge a$, is a Wiener process starting at time a at a non-random point x_0: $W_a = x_0$, we can not only consider integrals $\int_a^b f(t,\omega)\ dW_t$, but also $\int_c^b f(t,\omega)\ dW_t$, where $a < c < b$. Such integrals are defined as

$$\int_c^b f(t,\omega)\ dW_t = \int_a^b \begin{Bmatrix} 0, & a \le t \le c, \\ f(t,\omega), & c < t \le b \end{Bmatrix}\ dW_t. \tag{6.18}$$

The integrand here is just the kind of (mean-square) discontinuous random functions that we have considered above.

It is clear that for $a < c < b$

$$\int_a^b f(t,\omega)\ dW_t = \int_a^c f(t,\omega)\ dW_t + \int_c^b f(t,\omega)\ dW_t \tag{6.19}$$

(almost surely, of course – because our stochastic integrals are determined only *almost* uniquely).

In considering stochastic integral equations, we'll have to consider stochastic integrals $I(t) = I(t,\omega) = \int_{t_0}^t f(s,\omega)\ dW_s$ with variable upper limit $t \ge t_0$, being random functions.

Of course, $E\big(I(t)\big) = 0$, $E\big(I(t)^2\big) = \int_{t_0}^t E\big(f(s,\omega)^2\big)\ ds < \infty$ (see formulas (6.1), (6.2)). Also the random function $I(t)$ is mean-square continuous:

$$E\Big(\big(I(t') - I(t)\big)^2\Big) = E\Big(\big(\int_t^{t'} \Big(\text{or } \int_{t'}^t\Big)\ f(s,\omega)\ dW_s\big)^2\Big) \le \sup_s E\big(f(s,\omega)^2\big) \cdot |t' - t| \to 0 \tag{6.20}$$

as $t' - t \to 0$. And of course $I(t,\omega)$ is determined by the past (its value at the time point t is determined by the values of the Wiener process in the time interval $[t_0, t]$).

But what are the properties of *sample functions* of the random function $I(t) = I(t,\omega)$? Can we state that necessarily almost all sample functions are continuous?

It turns out that, in general, this question makes no sense. This is because for every t the value of the random function $I(t,\omega)$ at this time point, being defined as a stochastic integral, *is defined only up to some ω-set* (event) C_t *of zero probability*; since there are uncountably many such exceptional sets, their union $\bigcup_t C_t$ does not necessarily have zero probability, and may even have probability equal to 1.

Let us consider an example. Let W_t, $t \geq 0$, be a Wiener process starting at time 0 from the point $W_0 \equiv x_0 = 0$. Let us define the random function $I(t) = I(t,\omega)$, $t \geq 0$, by

$$I(t) = \begin{cases} 0, & t \neq 1 + |W_1|, \\ 1, & t = 1 + |W_1|. \end{cases} \tag{6.21}$$

This random function is determined by the past (the event $\{I(t) = 1\}$ is impossible for $t < 1$, so if we know W_s, $s \leq t$, we know whether this event has occurred (it has *not*, and even we don't need W_s, $s \leq t$, to know this); and for $t \geq 1$ we have: $\{I(t) = 1\} = \{|W_1| = t - 1\}$, and whether this event occurs is determined by the random variable W_1, $1 \leq t$. All sample functions $I(t,\,\omega)$ are *discontinuous* – namely, at the (random) point $t = 1 + |W_1(\omega)|$. At the same time, $I(t,\,\omega)$ is equal to the stochastic integral of the function $f(s,\,\omega) \equiv 0$:

$$I(t,\,\omega) = \int_0^t 0 \; dW_s. \tag{6.22}$$

Indeed, the stochastic integral is defined as a limit in mean; one of the versions of this stochastic integral is, of course, the identical 0. But every other random variable that is *equivalent to* 0, is also a version of the stochastic integral (6.22). And $I(t)$, for every $t \geq 0$, *is* equivalent to the identical 0:

$$P\{I_t \neq 0\} = P\{|W_1| = t - 1\} = 0, \tag{6.23}$$

because this probability is equal to the integral of the standard normal density over the set consisting of two points, $t - 1$ and $1 - t$: zero.

So we cannot ask whether the realization of the stochastic integral $\int_{t_0}^t f(s,\omega)\; dW_s$ is almost surely continuous in t: some *versions* of the stochastic integral $\int_{t_0}^t f(s,\omega)\; dW_s$ may be almost surely continuous, while some other discontinuous. What we can, and should, ask is whether *there exists* a version of the stochastic integral almost all of whose sample functions are continuous. The following theorem holds:

Theorem 6.1. *For a random function* $f(t,\omega)$, $t \geq t_0$, *of the kind considered above,* there exists *such a version of the stochastic integral*

$$I(t) = \int_{t_0}^t f(s,\omega)\; dW_s, \qquad t \geq t, \tag{6.24}$$

that almost surely its sample function $I(t,\omega)$ is continuous for $t \in [t_0, \infty)$.

For the time being, I am not giving the **proof** of this theorem; and I haven't decided yet whether I am going to give the proof later or not (after all, we did accept the continuity of the trajectories of the Wiener process without any proof). But the existence of a realization-continuous version of the stochastic integral (6.24) *can* be deduced from continuity of the Wiener sample functions, and the proof is simpler than the construction of the realization-continuous Wiener process. The first step in this proof is checking the fact for integrands being *step random functions.*

If $f(t,\omega)$ is a step random function with partition points $t_0 < t_1 < t_2 < \ldots$, we have:

$$\int_{t_0}^{t} f(s,\omega)\, dW_s = Y_1(\omega)\cdot(W_{t_1} - W_{t_0}) + Y_2(\omega)\cdot(W_{t_2} - W_{t_1}) + \ldots \\ + Y_{i-1}(\omega)\cdot(W_{t_{i-1}} - W_{t_{i-2}}) + Y_i(\omega)\cdot(W_t - W_{t_{i-1}}), \quad t_{i-1} \le t \le t_i. \tag{6.25}$$

Note that for t being equal to a partition point t_i, both formulas corresponding to both adjacent intervals $[t_{i-1}, t_i]$ (formula (6.25)) and $[t_i, t_{i+1}]$ (the same formula with i changed to $i+1$) yield exactly the same. And of course, formula (6.25) defines a continuous function of t changing in the interval $[t_{i-1}, t_i]$; so in this case $I_t(\omega)$ is continuous in t changing over all values ≥ 0 for *all* $\omega \in \Omega$, and not only *almost surely.*

To get the statement of the theorem in the general case, we have to perform a limit passage – and mean-square limit by itself does not provide any information about what happens with the sample functions.

Enough about this for the time being.

Now I want to introduce a concept I have not touched upon in the lectures before now: that of *stochastic differentials.*

Let W_t, $t \ge t_0$, be a Wiener process, $W_{t_0} \equiv x_0$; let X_t, $t \ge t_0$, be another stochastic process. We'll say that the process X_t *has a stochastic differential* if its trajectories are almost surely continuous, and there exist two functions $f(t,\omega)$ and $g(t,\omega)$, $t \ge t_0$, determined by the past (of the Wiener process) such that for every $t \ge t_0$ almost surely

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s,\omega)\, ds + \int_{t_0}^{t} g(s,\omega)\, dW_s. \tag{6.26}$$

(The first integral is understood as a usual one, the integration being performed for (almost) every $\omega \in \Omega$, and the second one, as a stochastic integral.) In this case we write the stochastic differential of the process X_t as

$$dX_t = f(t,\omega)\, dt + g(t,\omega)\, dW_t. \tag{6.27}$$

In the case of an n-dimensional Wiener process $\boldsymbol{W}_t = (W_t^1, \ldots, W_t^n)$, the differential is written as

$$dX_t = f(t,\omega)\, dt + \sum_{k=1}^{n} g_k(t,\omega)\, dW_t^k, \tag{6.28}$$

and this means, by definition, that almost surely

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s,\omega)\ ds + \sum_{k=1}^{n} \int_{t_0}^{t} g_k(s,\omega)\ dW_t^k. \tag{6.29}$$

In particular, it may be that every coordinate X_t^i of the r-dimensional stochastic process $\boldsymbol{X}_t$ has a stochastic differential:

$$dX_t^i = f_i(t,\omega)\, dt + \sum_{k=1}^{n} g_{ik}(t,\omega)\ dW_t^k, \qquad 1 \le i \le r, \tag{6.30}$$

where $\boldsymbol{f}(t,\omega) = \big(f_1(t,\omega), ..., f_r(t,\omega)\big)$ is a vector-valued random function, and $G(t,\omega) = \big(g_{ik}(t,\omega)\big)$ a matrix-valued one. If we rewrite all vectors as column vectors instead of row ones, the natural notation for (6.30) is

$$d\boldsymbol{X}_t = \boldsymbol{f}(t,\omega)\, dt + G(t,\omega)\ d\boldsymbol{W}_t. \tag{6.31}$$

Now let us consider one example in which we'll see what a stochastic integral is, and what strange properties it has.

For a one-dimensional Wiener process W_t, $t \ge a$, $W_a \equiv x_0 = \text{const}$ and a $b > a$ we'll find an expression for

$$\int_a^b W_t\ dW_t. \tag{6.32}$$

By definition, the integral (6.32) is

$$\underset{\max_{1\le t\le n}(t_i - t_{i-1})\to 0}{\text{l.i.m.}} W_{t_{i-1}} \cdot (W_{t_i} - W_{t_{i-1}}). \tag{6.33}$$

It turns out that the same summands are included in the following sum:

$$\begin{aligned}
(W_b - W_a)^2 &= \Big(\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})\Big)^2 \\
&= \sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} (W_{t_j} - W_{t_{j-1}}) \cdot (W_{t_i} - W_{t_{i-1}}) \\
&= \Sigma_{\mathfrak{T}} + 2\sum_{i=1}^{n} (W_{t_i} - W_a) \cdot (W_{t_i} - W_{t_{i-1}}) \\
&= \Sigma_{\mathfrak{T}} + 2\sum_{i=1}^{n} W_{t_i} \cdot (W_{t_i} - W_{t_{i-1}}) - 2W_a \cdot \sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}}).
\end{aligned} \tag{6.34}$$

It's time to stop writing equalities. (For $i = 1$, the sum $\sum_{j=1}^{i-1} = \sum_{j=1}^{0}$, containing no summands, is taken, as it should be, equal to 0.)

The sum $\Sigma_{\mathfrak{T}}$, as we know, converges in the mean square to $b-a$ as the partition $\mathfrak{T}$ becomes infinitely small, so we have:

$$\begin{aligned}\sum_{i=1}^{n} W_{t_i}\cdot(W_{t_i}-W_{t_{i-1}}) &= \frac{1}{2}(W_b-W_a)^2+W_a\cdot(W_b-W_a)-\frac{1}{2}\Sigma_{\mathfrak{T}} \\ &= \frac{1}{2}(W_b^2-W_a^2)-\frac{1}{2}\Sigma_{\mathfrak{T}} \to \frac{1}{2}(W_b^2-W_a^2)-\frac{1}{2}(b-a)\end{aligned} \tag{6.35}$$

as $\max_{1\le i\le n}(t_i-t_{i-1})\to 0$.

So we have (almost surely; or: the right-hand side is *one of the versions* of the stochastic integral in the left-hand side):

$$\int_a^b W_t\,dW_t = \frac{1}{2}(W_b^2-W_a^2)-\frac{1}{2}(b-a). \tag{6.36}$$

By the way, you can check that the expectation of our stochastic integral is equal to 0 – using the fact that $W_a^2\equiv x_0^2$, and the random variable W_b has the normal distribution with parameters $(x_0, b-a)$.

If a continuous (non-random) function $f(t)$ is Stieltjes integrable with respect to a continuous function $g(t)$, an integration by parts rule can be proved:

$$\int_a^b f(t)\,dg(t) = f(b)\cdot g(b)-f(a)\cdot g(a)-\int_a^b g(t)\,df(t). \tag{6.37}$$

In particular, if $f(t)$ is Stieltjes integrable with respect to the same $f(t)$, then

$$\int_a^b f(t)\,df(t) = f(b)^2-f(a)^2-\int_a^b f(t)\,df(t), \tag{6.38}$$

$$\int_a^b f(t)\,df(t) = \frac{1}{2}\big(f(b)^2-f(a)^2\big). \tag{6.39}$$

Since the right-hand side of (6.36) is definitely not equal to $\frac{1}{2}(W_b^2-W_a^2)$, we see that $W_t=W_t(\omega)$ is almost surely *not* Stieltjes integrable with respect to $W_t(\omega)$ (we knew that the trajectory of the Wiener process is not of bounded variation; but then, the condition of having bounded variation was only *sufficient* for integrating every continuous function to be possible). So the stochastic integral (in this case, and in general) cannot be understood as a Stieltjes integral.

If we take Stieltjes-like sums (6.6) with $t_i^*=t_i$, it turns out that their mean-square limit is equal to

$$\frac{1}{2}(W_b^2-W_a^2)+\frac{1}{2}(b-a). \tag{6.40}$$

This can be obtained using the fact that

$$\sum_{i=1}^{n} W_{t_i} \cdot (W_{t_i} - W_{t_{i-1}}) - \sum_{i=1}^{n} W_{t_{i-1}} \cdot (W_{t_i} - W_{t_{i-1}}) = \Sigma_{\mathfrak{T}}. \tag{6.41}$$

We can take a variable t in lieu of the right end b (denoting the integration variable s):

$$\int_a^t W_s \, dW_s = \frac{1}{2}(W_t^2 - W_a^2) - \frac{1}{2}(t-a), \tag{6.42}$$

$$W_t^2 = W_a^2 + \int_a^t 2W_s \, dW_s + \int_a^t 1 \, ds. \tag{6.43}$$

In the language of *stochastic differentials* this can be written as follows:

$$dW_t^2 = 2W_t \, dW_t + 1 \, dt. \tag{6.44}$$

We'll consider stochastic differential (in fact, integral) equations in the next lecture note.