Lecture 15. Characteristic functions.

Every characteristic function has the following properties:

$$f(0) = 1, \qquad |f(t)| \le 1, \quad t \in \mathbb{R}^1 \tag{15.1}$$

(because $|e^{it\xi}| \le 1$);

$$f(-t) = \overline{f(t)}, \qquad t \in \mathbb{R}^1 \tag{15.2}$$

(because $e^{i(-t)\xi} = e^{-it\xi} = \overline{e^{it\xi}}$); and

$$f(t) \text{ is continuous on } \mathbb{R}^1. \tag{15.3}$$

The last one because for every $\omega \in \Omega$

$$e^{it\xi(\omega)} \to e^{it_0\xi(\omega)} \qquad (t \to t_0), \tag{15.4}$$

and because of $|e^{it\xi}| \le$ (in fact $=$) 1 we can use Theorem 14.3.

The following theorem is, for characteristic functions, the same as Theorem 5.4 is for distribution functions:

Theorem 15.1. *Let* μ *and* ν *be two distributions on the real line* $\mathbb{R}^1$; *let* $f_\mu(t)$, $f_\nu(t)$ *be the corresponding characteristic functions*:

$$f_\mu(t) = \int_{-\infty}^{\infty} e^{itx}\,\mu(dx), \qquad f_\nu(t) = \int_{-\infty}^{\infty} e^{itx}\,\nu(dx). \tag{15.5}$$

If $f_\mu(t) = f_\nu(t)$ *for all* $t \in \mathbb{R}^1$, *then* $\mu = \nu$.

We are going to prove this theorem, which shows that we can operate with distributions using their characteristic functions – but not right now.

The obvious properties (15.1) – (15.3) are not enough for a function $f(t)$, $-\infty < t < \infty$, to be a characteristic function: no analog of Theorem 5.5 with these properties is true.

We had the formula

$$f(t) = \sum_{k=-\infty}^{\infty} p(kh) \cdot e^{ikht} \tag{14.21}$$

for the characteristic function of a discrete random variable taking values that are multiples of h; and the formula

$$p(kh) = \frac{1}{2\pi/h} \int_{-\pi/h}^{\pi/h} e^{-ikht} f(t)\,dt \tag{14.22}$$

for the Fourier coefficients, which are the values of the probability mass function.

The formula for continuous distributions corresponding to the first of these:

$$f(t) = \int_{-\infty}^{\infty} p(x) \cdot e^{itx} \, dx \tag{15.6}$$

where $p(x)$ is no longer the probability mass function, but the density. The integral (15.6) converges absolutely.

This is what we call Fourier transform.

There is a theory of Fourier transforms that is parallel to that of Fourier series. In particular, there are formulas expressing the inverse Laplace transforms, similar to (14.22). Let me formulate one result of this kind (its formulation – if not its proof – is simple):

Theorem 15.2. *If* $p(x)$ *is integrable over* $(-\infty, \infty)$, *and its Fourier transform expressed by formula* (15.6) *also is* (*Lebesgue-*) *integrable*:

$$\int_{-\infty}^{\infty} |f(t)| \, dt < \infty, \tag{15.7}$$

then for almost all $x \in (-\infty, \infty)$ (with respect to the Lebesgue measure)

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t) \, dt. \tag{15.8}$$

Of course, since a density is not determined uniquely, we could not expect that (15.8) should hold *everywhere*. Another formulation is possible:

Then there exists a continuous version of the density $p(x)$, *and it is given by formula* (15.8).

This way we can find the distribution whose characteristic function is given by

$$f(t) = e^{-|t|}, \qquad -\infty < t < \infty \tag{15.9}$$

(draw the graph of this function).

The function (15.9) is certainly integrable; let us apply formula (15.8):

$$\begin{aligned} p(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \cdot e^{-|t|} \, dt = \frac{1}{2\pi} \Big[\int_{0}^{\infty} e^{-itx-t} \, dt + \int_{-\infty}^{0} e^{-itx+t} \, dt \Big] \\ &= \frac{1}{2\pi} \Big[\big[\frac{e^{-itx-t}}{-1-ix} \big]_0^{\infty} + \big[\frac{e^{-itx+t}}{1-ix} \big]_{-\infty}^{0} \Big] = \frac{1}{2\pi} \big[\frac{1}{1+ix} + \frac{1}{1-ix} \big] = \frac{\pi^{-1}}{1+x^2}. \end{aligned} \tag{15.10}$$

This is the density of the standard *Cauchy* distribution.

We could have obtained the fact that the characteristic function of the Cauchy distribution is $e^{-|t|}$ using formula (15.6) and evaluating the integral $\int_{-\infty}^{\infty} \frac{\pi^{-1}}{1+x^2} \cdot e^{itx} \, dx$; but this would be more difficult: the easiest way to handle this integral is using the theory of functions of complex variable.

After these examples, let us go to the general theory.

Proof of Theorem 15.1. We have to prove that, given $f_\mu(t) = f_\nu(t)$, $t \in (-\infty, \infty)$, for every Borel set $C \subseteq \mathbb{R}^1$ we have:

$$\mu(C) = \nu(C). \tag{15.11}$$

This formula can be rewritten as

$$\int_{-\infty}^{\infty} g(x)\ \mu(dx) = \int_{-\infty}^{\infty} g(x)\ \nu(dx) \tag{15.12}$$

(for the function $g(x) = I_C(x)$). Let us denote by $\mathfrak{G}$ the class of all bounded measurable (complex-valued) functions g for which the equality (15.1)2 holds.

By the condition $f_\mu(t) = f_\nu(t)$, $t \in (-\infty, \infty)$, for every real t the function $g_t \in \mathfrak{G}$, where

$$g_t(x) = e^{itx}. \tag{15.13}$$

Lemma 15.1. *If* $g_1, \ldots, g_n \in \mathfrak{G}$, *then an arbitrary linear combination of these functions with complex coefficient also belongs to* $\mathfrak{G}$.

The **proof** is so simple that we omit it.

Let us introduce a notation: $g_n \rightharpoonup g$ will mean that $g_n(x) \to g(x)$ $(n \to \infty)$ for every x, and there exists a constant $C < \infty$ such that $|g_n(x)| \le C$ for all n and x.

Lemma 15.2. *If* $g_1, \ldots, g_n, \ldots \in \mathfrak{G}$ *and* $g_n \rightharpoonup g$, *then* $g \in \mathfrak{G}$.

The **proof** follows immediately from Theorem 14.3:

$$\int_{-\infty}^{\infty} g(x)\ \mu(dx) = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\ \mu(dx) = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\ \nu(dx) = \int_{-\infty}^{\infty} g(x)\ \nu(dx). \tag{15.14}$$

Lemma 15.3. *Let* $g(x)$ *be a twice continuously differentiable periodic function on* $(-\infty, \infty)$. *Then* $g \in \mathfrak{G}$.

Proof. The function $g(x)$ is represented as the sum of its Fourier series, which converges absolutely and uniformly:

$$g(x) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{2k\pi i x/T}, \tag{15.15}$$

where T is the period, and the Fourier coefficients are given by

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} e^{-2k\pi i x/T} g(x)\ dx \tag{15.16}$$

(uniform absolute convergence follows from integration by parts: for $k \neq 0$ we have:

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-2k\pi ix/T} g(x)\;dx \\ &= \frac{1}{T}\cdot\Big[\big[\frac{e^{-2k\pi ix/T}}{-2k\pi i/T}\cdot g(x)\big]_{-T/2}^{T/2} - \int_{-T/2}^{T/2}\frac{e^{-2k\pi ix/T}}{-2k\pi i/T}\cdot g'(x)\;dx\Big] \\ &= -\frac{1}{T}\int_{-T/2}^{T/2}\frac{e^{-2k\pi ix/T}}{-2k\pi i/T}\cdot g'(x)\;dx \\ &= \frac{1}{T}\int_{-T/2}^{T/2}\frac{e^{-2k\pi ix/T}}{(-2k\pi i/T)^2}\cdot g''(x)\;dx, \end{aligned} \tag{15.17}$$

and $|c_k| \leq \dfrac{T}{4k^2\pi^2}$, the series $\sum_k |c_k \cdot e^{2k\pi ix/T}|$ being dominated by a convergent series $\sum_k \dfrac{\text{const}}{k^2}$).

That the Fourier series (15.15) *converges* (uniformly and absolutely), we have proved above; that it converges *to* $g(x)$ is not so simple; but this is a fact from Analysis, and we are using it without proof.

So we have $g_n \rightharpoonup g$, where

$$g_n(x) = \sum_{k=-n}^{n} c_k \cdot e^{2k\pi ix/T} \tag{15.18}$$

(there is convergence for every x, and all these functions are dominated by the constant $C = |c_0| + \sum_{k\neq 0}\dfrac{1}{k^2}\cdot\dfrac{\int_{-T/2}^{T/2}|g''(x)|\;dx}{T} < \infty$). The functions g_n belong to $\mathfrak{G}$ because they are linear combinations of the functions $g_{2k\pi/T}$. Therefore $g \in \mathfrak{G}$.

Lemma 15.4. *Let $g(x)$ be a twice continuously differentiable function on the real line, such that $g(x) = 0$ for all x such that $|x| \geq a$, where a is some positive number. Then $g \in \mathfrak{G}$.*

Proof. For all $T > 2a$, let us define a function g_T by

$$g_T(x) = g(x - kT) \qquad \text{for } |x - kT| \leq T/2, \quad k = 0, \pm1, \pm2, \pm3, \ldots \tag{15.19}$$

(we extend the function $g(x)$ T-periodically beyond the interval $[-T/2, T/2]$; make a picture).

Since g_T is smooth and periodic, it belongs to $\mathfrak{G}$. All these functions are dominated by the constant $\max_x |g(x)| < \infty$; and for every $x \in (-\infty, \infty)$ we have:

$$\lim_{T\to\infty} g_T(x) = g(x). \tag{15.20}$$

It is true that this convergence is not uniform; but Lemma 15.2 does not require *uniform* convergence, and from $g_T \rightharpoonup g$ we get $g \in \mathfrak{G}$.

Lemma 15.5. *For every pair of real numbers* $-\infty < a < b < \infty$ *we have*:

$$I_{(a,\,b]} \in \mathfrak{G} \quad - \tag{15.21}$$

which means that

$$\mu(a,\,b] = \nu(a,\,b]. \tag{15.22}$$

Proof. For every positive $\varepsilon < b-a$ let us take

$$g_\varepsilon(x) = \begin{cases} 0, & x \le a, \\ 6\big(\frac{x-a}{\varepsilon}\big)^5 - 15\big(\frac{x-a}{\varepsilon}\big)^4 + 10\big(\frac{x-a}{\varepsilon}\big)^3, & a \le x \le a+\varepsilon, \\ 1, & a+\varepsilon \le x \le b, \\ 1 - 6\big(\frac{x-b}{\varepsilon}\big)^5 + 15\big(\frac{x-b}{\varepsilon}\big)^4 - 10\big(\frac{x-b}{\varepsilon}\big)^3, & b \le x \le b+\varepsilon, \\ 0, & x \ge b. \end{cases} \tag{15.23}$$

Make a picture of the graph of this function; check that $g_\varepsilon(a) = g_\varepsilon(b+\varepsilon) = 0$, $g_\varepsilon(a+\varepsilon) = g(b) = 1$, $g'_\varepsilon(a) = g'_\varepsilon(a+\varepsilon) = g'_\varepsilon(b) = g'_\varepsilon(b+\varepsilon) = g''_\varepsilon(a) = g''_\varepsilon(a+\varepsilon) = g''_\varepsilon(b) = g''_\varepsilon(b+\varepsilon) = 0$, and this function is twice continuously differentiable. All functions $g_\varepsilon(x)$ belong to $\mathfrak{G}$ (by Lemma 15.4); they all are dominated by the constant

$$\max_{0 \le u \le 1} |6u^5 - 15u^4 + 10u^3| \tag{15.24}$$

(in fact, this maximum is equal to 1); and for every $x \in (-\infty,\,\infty)$ we have $\lim_{\varepsilon \to 0^+} g_\varepsilon(x) = g(x)$, i.e., $g_\varepsilon \rightharpoonup g$. So $g \in \mathfrak{G}$.

This means that for every $-\infty < a < b < \infty$, we have:

$$\mu(a,\,b] = \int_{-\infty}^{\infty} I_{(a,\,b]}(x)\;\mu(dx) = \nu(a,\,b] = \int_{-\infty}^{\infty} I_{(a,\,b]}(x)\;\nu(dx). \tag{15.25}$$

In terms of distribution functions, this is expressed as follows:

$$F_\mu(b) - F_\mu(a) = \mu(-\infty,\,b] - \mu(-\infty,\,a] = F_\nu(b) - F_\nu(a) = \nu(-\infty,\,b] - \nu(-\infty,\,a]. \tag{15.26}$$

Let us take $a \to -\infty$ in (15.26):

$$F_\mu(b) = \lim_{a \to -\infty} [F_\mu(b) - F_\mu(a)] = \lim_{a \to -\infty} [F_\nu(b) - F_\nu(a)] = F_\nu(b), \tag{15.27}$$

and this for an arbitrary $b \in (-\infty,\,\infty)$. The distribution functions corresponding to the distributions μ and ν coincide, and so by Theorem 5.4 the distributions coincide: $\mu = \nu$.