Lecture 19. Weak convergence and characteristic functions.

Proof of Theorem 18.2: We want to prove that if x_0 is a continuity point of G, then for every positive ε there exists a natural n_0 such that for every $n \geq n_0$

$$|F_n(x_0) - G(x_0)| < \varepsilon. \tag{19.1}$$

Let us fix an arbitrary positive ε.

Since x_0 is a continuity point of G, there exists a positive δ such that for $|x - x_0| \leq \delta$ we have $|G(x) - G(x_0)| < \varepsilon/2$.

We have: $F_n(x_0) = \int_{-\infty}^{\infty} I_{(-\infty,\, x_0]}(x)\ \mu_n(dx)$, $G(x_0) = \int_{-\infty}^{\infty} I_{(-\infty,\, x_0]}(x)\ \nu(dx)$; the function $I_{(-\infty,\, x_0]}(x)$ is bounded, but not continuous. Let us "sandwich" it between two continuous functions: namely, let us defined two bounded continuous functions, $\underline{f}(x)$ and $\overline{f}(x)$:

$$\underline{f}(x) = \begin{cases} 1, & x \leq x_0 - \delta, \\ 1 - \dfrac{x - (x_0 - \delta)}{\delta}, & x_0 - \delta \leq x \leq x_0, \\ 0, & x \geq x_0, \end{cases} \tag{19.2}$$

$$\overline{f}(x) = \begin{cases} 1, & x \leq x_0, \\ 1 - \dfrac{x - x_0}{\delta}, & x_0 \leq x \leq x_0 + \delta, \\ 0, & x \geq x_0 + \delta \end{cases} \tag{19.3}$$

(make a picture and see that these functions are indeed bounded and continuous).

We have, for the indicator functions of the intervals $(-\infty,\, x_0 - \delta]$, $(-\infty,\, x_0]$, and $(-\infty,\, x_0 + \delta]$:

$$I_{(-\infty,\, x_0-\delta]}(x) \leq \underline{f}(x) \leq I_{(-\infty,\, x_0]}(x) \leq \overline{f}(x) \leq I_{(-\infty,\, x_0+\delta]}(x) \tag{19.4}$$

(the same picture of yours).

Because of $\mu_n \to_w \nu$, there exists a natural n_0 such that for $n \geq n_0$ we have:

$$\begin{aligned} \Big| \int_{-\infty}^{\infty} \underline{f}(x)\ \mu_n(dx) - \int_{-\infty}^{\infty} \underline{f}(x)\ \nu(dx) \Big| &< \varepsilon/2, \\ \Big| \int_{-\infty}^{\infty} \overline{f}(x)\ \mu_n(dx) - \int_{-\infty}^{\infty} \overline{f}(x)\ \nu(dx) \Big| &< \varepsilon/2. \end{aligned} \tag{19.5}$$

So we have (for $n \ge n_0$):

$$
\begin{aligned}
G(x_0) - \varepsilon < G(x_0 - \delta) - \frac{\varepsilon}{2} &= \int_{-\infty}^{\infty} I_{(-\infty,\, x_0 - \delta]}(x)\ \nu(dx) - \frac{\varepsilon}{2} \\
\le \int_{-\infty}^{\infty} \underline{f}(x)\ \nu(dx) - \frac{\varepsilon}{2} &< \int_{-\infty}^{\infty} \underline{f}(x)\ \mu_n(dx) \le \int_{-\infty}^{\infty} I_{(-\infty,\, x_0]}(x)\ \mu_n(dx) = F_n(x_0) \\
\le \int_{-\infty}^{\infty} \overline{f}(x)\ \mu_n(dx) &< \int_{-\infty}^{\infty} \underline{f}(x)\ \nu(dx) + \frac{\varepsilon}{2} \le \int_{-\infty}^{\infty} I_{(-\infty,\, x_0+\delta]}(x)\ \nu(dx) + \frac{\varepsilon}{2} \\
&= G(x_0 + \delta) + \frac{\varepsilon}{2} < G(x_0) + \varepsilon.
\end{aligned}
\tag{19.6}
$$

Check every equality and every inequality in this chain, out of which we need only $G(x_0) - \varepsilon < F_n(x_0) < G(x_0) + \varepsilon$.

The theorem is proved.

If $\mu_n \to_w \nu$, can the corresponding sequence of distribution functions $F_n(x)$ converge to $G(x) = \nu(-\infty, x]$ also at (some of) discontinuity points of G? Or is it that necessarily $F_n(x) \to G(x)$ for *all* x, only we did not include this in our Theorem 18.2 because it is too complicated (or even an open mathematical problem)? – It turns out that all cases are possible.

Example: $\xi_n = 1/n$ with probability $(n-1)/n$, $\xi_n = n$ with probability $1/n$; and $\eta \equiv 0$. The distribution function $F_n(x)$ of the random variable ξ_n is given by

$$
F_n(x) = \begin{cases} 0, & x < 1/n, \\ (n-1)/n, & 1/n \le x < n, \\ 1, & x \ge n, \end{cases} \tag{19.7}
$$

and

$$
G(x) = P\{\eta \le x\} = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases} \tag{19.8}
$$

It is easy to see that for every bounded continuous function $f(x)$

$$
\int_{-\infty}^{\infty} f(x)\ \mu_{\xi_n}(dx) = f\Big(\frac{1}{n}\Big) \cdot \frac{n-1}{n} + f(n) \cdot \frac{1}{n} \to f(0) \cdot 1 = \int_{-\infty}^{\infty} f(x)\ \mu_\eta(dx), \tag{19.9}
$$

$\mu_{\xi_n} \to_w \mu_\eta$.

As for the distribution functions,

$$
\lim_{n\to\infty} F_n(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases} \tag{19.10}
$$

This limit coincides with $G(x)$ for $x \ne 0$, which are the continuity points of the function G, but not for $x = 0$.

Another example: same with $\xi_n = -1/n$ with probability $(n-1)/n$, and $\xi_n = n$ with probability $1/n$. Here $\lim_{n\to\infty} F_{\xi_n}(x) = G(x)$ for *all* real x.

And if $\xi_n = (-1)^n/n$ with probability $(n-1)/n$, and $P\{\xi_n = n\} = 1/n$, the limit $\lim_{n\to\infty} F_{\xi_n}(x)$ *does not exist* for $x = 0$ (but of course it exists and is equal to $G(x)$ for all $x \neq 0$, which are continuity points).

Theorem 19.1. *Let $F_1(x)$, $F_2(x)$, ..., $F_n(x)$, ... be a sequence of distribution functions; let K_0 be a countable set that is dense in $\mathbb{R}^1$. Then there exists a nondecreasing function $G(x)$, $-\infty < x < \infty$, and a subsequence $n_k \to \infty$ such that*

$$F_{n_k}(x) \to G(x) \qquad (k \to \infty) \tag{19.11}$$

for all $x \in K_0$.

Proof. Let $K_0 = \{x_1, x_2, x_3, ..., x_n, ...\}$. Take the sequence of numbers $F_n(x_1)$, $n = 1, 2, 3, ...$. These numbers all are in $[0, 1]$, so there exists a subsequence

$$\{n_1^1, n_2^1, ..., n_k^1, ...\} \subseteq \{1, 2, 3, ..., n, ...\}, \tag{19.12}$$

$n_1^1 < n_2^1 < ... < n_k^1 < ...$, such that $F_{n_k^1}(x_1) \to g_1$ $(k \to \infty)$.

Then we take the sequence $F_{n_k^1}(x_2)$, $k = 1, 2, 3, ...$; it is bounded, so we can choose

$$\{n_1^2, n_2^2, ..., n_k^2, ...\} \subseteq \{n_1^1, n_2^1, ..., n_k^1, ...\}, \tag{19.13}$$

$n_1^2 < n_2^2 < ... < n_k^2 < ...$, so that $F_{n_k^2}(x_2) \to g_2$ $(k \to \infty)$. Proceeding like this, we obtain a non-increasing sequence of sequences $n_1^j < n_2^j < ... < n_k^j < ...$ such that $F_{n_k^j}(x_j) \to g_j$ $(k \to \infty)$. Now we take the *diagonal sequence* $n_k = n_k^k$, $k = 1, 2, 3, ...$. This sequence is a subsequence of $\{n_1^1, n_2^1, ..., n_k^1, ...\}$, so $\lim_{k\to\infty} F_{n_k}(x_1) = g_1$; it is a subsequence of the sequence $\{n_1^2, n_2^2, ..., n_k^2, ...\}$ starting with $n_2 = n_2^2$ ($n_1 = n_1^1$ may not be in $\{n_1^2, n_2^2, ..., n_k^2, ...\}$), so $\lim_{k\to\infty} F_{n_k}(x_2) = g_2$; etc. So we get for all $x_j \in K_0$:

$$\lim_{k\to\infty} F_{n_k}(x_j) = g_j. \tag{19.14}$$

Now we define the function $G(x)$, $-\infty < x < \infty$. First we define this function for $x \in K_0$ by

$$G(x_j) = g_j. \tag{19.15}$$

On this set the function $G(x)$ is nondecreasing, which means that if $x_j < x_s$, we have

$$G(x_j) \leq G(x_s). \tag{19.16}$$

This follows from the fact that the functions $F_{n_k}(x)$ are non-decreasing, so

$$F_{n_k}(x_j) \leq F_{n_k}(x_s), \tag{19.17}$$

and from

$$G(x_j) = \lim_{k\to\infty} F_{n_k}(x_j), \qquad G(x_s) = \lim_{k\to\infty} F_{n_k}(x_s). \tag{19.18}$$

Now, for every monotone function on the set K_0 one-sided limits exist at every point; in particular, for every $x \in (-\infty, \infty)$ there exists the right-hand limit

$$\lim_{x_j \to x^+} G(x_j). \tag{19.19}$$

(this limit is defined as the number L such that for every positive ε there exists a positive δ such that for all $x_j \in K_0$ belonging to the interval $(x, x+\delta)$ we have $|G(x_j) - L| < \varepsilon$; since the set K_0 is dense, such x_j's exist for every positive δ). Now we define the function $G(x)$ for all x outside K_0 by

$$G(x) = \lim_{x_j \to x^+} G(x_j), \qquad x \notin K_0. \tag{19.20}$$

Now the function $G(x)$ is defined for all $x \in (-\infty, \infty)$. Let us prove that this is a nondecreasing function:

$$G(x) \leq G(y) \qquad \text{for } x < y. \tag{19.21}$$

We have to consider four cases: $x, y \in K_0$; $x \in K_0$, $y \notin K_0$; $x \notin K_0$, $y \in K_0$; and x, $y \notin K_0$.

The first case is just (19.16). The second case: Let $x = x_j < y \notin K_0$; we have to prove that

$$G(x_j) \leq \lim_{x_s \to y^+} G(x_s). \tag{19.22}$$

This follows from $x_j < y < x_s$, $G(x_j) \leq G(x_s)$.

The third case: $K_0 \not\ni x < y = x_s$, we have to prove

$$\lim_{x_j \to x^+} G(x_j) \leq G(x_s); \tag{19.23}$$

this follows from $x_j < x + \delta < y = x_s$ for $0 < \delta < y - x$. Finally, our statement in the fourth case means that

$$\lim_{x_j \to x+} G(x_j) \leq \lim_{x_s \to y+} G(x_s), \tag{19.24}$$

and follows from $x_j < x + \delta < y < x_s$, which holds for $0 < \delta < y - x$.

The theorem is proved.

However the limiting function $G(x)$ is not necessarily a distribution function: it is nondecreasing, all its values are between 0 and 1; but it may be not continuous from the right, and $\lim_{x\to\infty} G(x) = 1$, $\lim_{x\to-\infty} G(x) = 0$ do not necessarily hold. It turns out that we can do something with the right-continuity by taking another dense set $K \subset \mathbb{R}^1$ (see our next theorem); but nothing, generally, with the limits at $\pm\infty$, as the following example shows:

Let $F_n(x)$ be the distribution function of the normal distribution with parameters $(0, n)$ (with expectation 0 and variance n):

$$F_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi n}}\, e^{-u^2/2n}\, du. \tag{19.25}$$

Making a substitution $u/\sqrt{n} = v$, $u = \sqrt{n} \cdot u$, $du = \sqrt{n} \cdot dv$, we come to

$$F_n(x) = \int_{-\infty}^{x/\sqrt{n}} \frac{1}{\sqrt{2\pi}}\, e^{-v^2/2}\, dv. \tag{19.26}$$

Now take the limit as $n \to \infty$: for all real x

$$G(x) = \lim_{n\to\infty} F_n(x) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}}\, e^{-v^2/2}\, dv = \frac{1}{2}. \tag{19.27}$$

Definitely *not* a distribution function!

Theorem 19.2. *Under the conditions of the previous theorem, there exist a nondecreasing right-continuous function* $\tilde{G}(x)$, $-\infty < x < \infty$, *a dense set* $K \subseteq (-\infty, \infty)$, *and a sequence* $n_k \to \infty$ *such that*

$$\lim_{k\to\infty} F_{n_k}(x) = \tilde{G}(x) \qquad \text{for all } x \in K. \tag{19.28}$$

Proof (not finished in the lecture, but put in the present lecture note). Since the function $G(x)$ is monotone, it has one-sided limits at every point; its discontinuity points are just jumps, and there are at most a countable number of them.

For every $x \in (-\infty, \infty)$ define $\tilde{G}(x)$ as the right-hand limit of the function G at this point:

$$\tilde{G}(x) = \lim_{y\to x^+} G(y) \tag{19.29}$$

(note that this right-hand limit is the same as the limit $\lim\limits_{K_0 \ni x_j \to x^+} G(x_j)$ that was used in formula (19.20), but there it was taken only for $x \notin K_0$, while here it is for *all* $x \in \mathbb{R}^1$).

The function $\tilde{G}$ is, of course, non-decreasing. Let us prove that it is continuous from the right. The formula (19.29) means that for every positive ε there exists a positive δ such that for all $y \in (x, x+\delta)$

$$G(y) - \tilde{G}(x) < \varepsilon \tag{19.30}$$

(I am not writing $|G(y) - \tilde{G}(x)|$ because anyway $G(y) - \tilde{G}(x) \geq 0$). Now, right continuity means that for every $\varepsilon > 0$ there exists a positive δ such that for all $z \in (x, x+\delta)$

$$\tilde{G}(z) - \tilde{G}(x) < \varepsilon. \tag{19.31}$$

But $\tilde{G}(z) = \lim_{w\to z^+} G(w)$, and because the function G is nondecreasing, this right-hand limit is not greater than $G(w)$ for every $w > z$. Among such w's there are such that belong to the interval $(x, x+\delta)$ (draw a picture); so by (19.30) $G(w) < \tilde{G}(x) + \varepsilon$, and $\tilde{G}(z) = \lim_{w\to z^+} G(w) < \tilde{G}(x) + \varepsilon$: (19.31) holds.

Note that if the function G is continuous at a point x, we have:

$$\tilde{G}(x) = \lim_{y\to x^+} G(y) = G(x). \tag{19.32}$$

Let us take as the set K the set of all continuity points of the function G (which is, of course, the same as the set of continuity points of the function $\tilde{G}$). We are going to prove that $\lim_{k\to\infty} F_{n_k}(x) = \tilde{G}(x)$ for $x \in K$ (n_k is the subsequence of Theorem 19.1). The proof is similar to that of Theorem 18.2.

Let us take a point $x_0 \in K$ (i. e. a point at which the function G is continuous). We have to prove that for every positive ε there is a natural k_0 such that for every $k \geq k_0$

$$|F_{n_k}(x_0) - \tilde{G}(x_0)| < \varepsilon; \tag{19.33}$$

or, which is the same,

$$G(x_0) - \varepsilon < F_{n_k}(x_0) < G(x_0) + \varepsilon \tag{19.34}$$

(we replaced $\tilde{G}(x_0)$ with $G(x_0)$ because of (19.32)).

Let us take a positive δ so that

$$G(x_0 + \delta) < G(x_0) + \varepsilon/2, \qquad G(x_0 - \delta) > G(x_0) - \varepsilon/2. \tag{19.35}$$

Take a point $x_s \in K_0$ so that $x_s \in (x_0,\, x_0 + \delta)$, and a point $x_j \in K_0$ such that $x_j \in (x_0 - \delta,\, x_0)$; then

$$G(x_s) \leq G(x_0 + \delta) < G(x_0) + \varepsilon/2, \qquad G(x_j) \geq G(x_0 - \delta) > G(x_0) - \varepsilon/2. \tag{19.36}$$

We have: $\lim_{k\to\infty} F_{n_k}(x_j) = G(x_j)$, $\lim_{k\to\infty} F_{n_k}(x_s) = G(x_s)$; so there exists a k_0 such that for every $k \geq k_0$

$$|F_{n_k}(x_j) - G(x_j)| < \varepsilon/2, \qquad |F_{n_k}(x_s) - G(x_s)| < \varepsilon/2. \tag{19.37}$$

From (19.36) and (19.37) we get:

$$F_{n_k}(x_s) < G(x_0) + \varepsilon, \qquad F_{n_k}(x_j) > G(x_0) - \varepsilon. \tag{19.38}$$

But the function F_{n_k} is non-decreasing, so we obtain:

$$G(x_0) - \varepsilon < F_{n_k}(x_j) \leq F_{n_k}(x_0) \leq F_{n_k}(x_s) < G(x_0) + \varepsilon, \tag{19.39}$$

and (19.33) is proved.