Lecture 28. Law of large numbers for dependent random variables.

Theorem 28.1. *Let* ξ_0, ξ_1, ξ_2, ..., ξ_n, ... *be a sequence of random variables with values in a measurable space* $(X, \mathcal{X})$; *and* $f(x)$ *a bounded measurable function on this space.*

If $\alpha^*(\mathcal{F}_{\le n}, \mathcal{F}_{\ge m}) \le \beta(m-n)$, $\beta(k) \to 0$ $(k \to \infty)$, *then weak law of large numbers holds for the sequence of random variables* $\eta_i = f(\xi_i)$, $i = 0, 1, 2, ..., n, ...$:

$$\frac{\eta_0 + \eta_1 + ... + \eta_{n-1}}{n} - \frac{E(\eta_0 + \eta_1 + ... + \eta_{n-1})}{n} \to_P 0 \quad (n \to \infty). \tag{28.1}$$

The **proof** is similar to that of Chebyshev's Theorem (Theorem 13.7). We have:

$$\mathrm{Var}\big(\frac{\eta_0 + \eta_1 + ... + \eta_{n-1}}{n}\big) = \frac{1}{n^2} \sum_{i,\, j=0}^{n-1} \mathrm{Cov}(\eta_i,\, \eta_j) \tag{28.2}$$

(remember that "on the diagonal", for $j = i$, it is just the sum of the variances). Let us rewrite this sum so that it does not involve pairs $(i,\, j)$ with $i > j$:

$$\frac{1}{n^2}\big[\sum_{i=0}^{n-1} \mathrm{Cov}(\eta_i,\, \eta_i) + 2 \sum_{0 \le i < j < n} \mathrm{Cov}(\eta_i,\, \eta_j)\big] \le \frac{2}{n^2} \sum_{0 \le i \le j < n} \mathrm{Cov}(\eta_i,\, \eta_j) \tag{28.3}$$

(the last sum is just shorter to write than writing two previous sums). Let us change the summation variables in the last sum: instead of i, j, let it be $k = j - i$ and i: the right-hand side of (28.3) is equal to

$$\sum_{k=0}^{n-1} \big[\sum_{i=0}^{n-1-k} \mathrm{Cov}(\eta_i,\, \eta_{i+k})\big]. \tag{28.4}$$

The random variable $\eta_i = f(\xi_i)$ is measurable with respect to the σ-algebra $\mathcal{F}_{\le i}$, and η_{i+k} with respect to $\mathcal{F}_{\le i+k}$. By the definition of the coefficient of dependence, we have:

$$\mathrm{Cov}(\eta_i,\, \eta_{i+k}) \le C^2 \cdot \alpha^*(\mathcal{F}_{\le i},\, \mathcal{F}_{\ge i+k}) \le C^2 \cdot \beta(k), \tag{28.5}$$

where C is a constant such that $|f(x)| \le C$ for all x (the coefficient α^* was defined using random variables bounded by 1 in absolute value; of course, if one of them is bounded by a constant C, and the other one by D, the bound for the covariance should be multiplied by $C \cdot D$).

So the variance (28.2) does not exceed

$$\frac{2C^2}{n^2} \sum_{k=0}^{n-1} \big[\sum_{i=0}^{n-1-k} \beta(k)\big] = \frac{2C^2}{n^2} \sum_{k=0}^{n-1} (n-k)\beta(k) \le \frac{2C^2}{n^2} \sum_{k=0}^{n-1} n\beta(k) = 2C^2\, \frac{\sum_{k=0}^{n-1} \beta(k)}{n}. \tag{28.6}$$

Let us prove that this goes to 0 as $n \to \infty$. This means that for every positive ε there exists a natural number n_0 such that the expression (28.6) is smaller than ε for $n \geq n_0$.

We haven't used the fact that $\beta(k) \to 0$ $(k \to \infty)$ yet. Let us use it now: for every positive ε there exists a natural k_0 such that for $k \geq k_0$

$$\beta(k) < \frac{\varepsilon}{4C^2}. \tag{28.7}$$

Dividing the sum in the right-hand side into two: over $k < k_0$ and over $k \geq k_0$, we obtain, for $n \geq k_0$ that the variance (28.2) is not greater than

$$\begin{aligned} \frac{2C^2}{n}\Big[\sum_{k=0}^{k_0-1}\beta(k) + \sum_{k=k_0}^{n-1}\beta(k)\Big] &< \frac{2C^2}{n}\Big[\sum_{k=0}^{k_0-1}\beta(k) + \sum_{k=k_0}^{n-1}\frac{\varepsilon}{4C^2}\Big] \\ &< \frac{2C^2}{n}\Big[\sum_{k=0}^{k_0-1}\beta(k) + \frac{n\varepsilon}{4C^2}\Big] = 2C^2\frac{\sum_{k=0}^{k_0-1}\beta(k)}{n} + \frac{\varepsilon}{2}. \end{aligned} \tag{28.8}$$

The first term in the right-hand side goes to 0 as $n \to \infty$, so it is less than $\dfrac{\varepsilon}{2}$ for $n \geq n_0$ (where we did not forget to take $n_0 \geq k_0$ so that $n \geq n_0 \ \Rightarrow\ n \geq k_0$). From this we see that the variance (28.2) goes to 0 as $n \to \infty$.

The rest of the proof follows that of Chebyshev's Theorem: we use the classical Chebyshev's inequality: for every $\varepsilon > 0$

$$\begin{aligned} P\Big\{\Big|\frac{\eta_0+\eta_1+\ldots+\eta_{n-1}}{n} - \frac{E(\eta_0+\eta_1+\ldots+\eta_{n-1})}{n}\Big| \geq \varepsilon\Big\} \\ \leq \frac{\operatorname{Var}\big((\eta_0+\eta_1+\ldots+\eta_{n-1})/n\big)}{\varepsilon^2} \to 0 \quad (n\to\infty). \end{aligned} \tag{28.9}$$

Theorem 28.1 holds also for different functions: $\eta_i = f_i(\xi_i)$ if all these functions are bounded by the same constant: $|f_i(x)| \leq C$ for all $i = 0, 1, 2, \ldots, n, \ldots$.

Using Theorems 28.1 and 27.2, we can prove the following

Theorem 28.2. *Let* $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ *be a finite ergodic Markov chain*; $\eta_i = f(\xi_i)$. *Then*

$$\frac{\eta_0+\eta_1+\ldots+\eta_{n-1}}{n} \to_P a, \tag{28.10}$$

where the constant a *is defined by*

$$a = \sum_{y\in X} p_y \cdot f(y), \tag{28.11}$$

p_y *being the limiting probabilities*: $p_y = \lim_{k\to\infty} p_{xy}^{(k)}$, $x, y \in X$.

(We couldn't claim that the limit of the arithmetic mean of the first n random variables exists under the condition of Theorem 28.1, but under more restrictive conditions of Theorem 28.2 we can.)

Proof. In addition to what Theorems 27.2 and 28.1 give us, it is enough to prove that

$$\lim_{n\to\infty} \frac{E\eta_0 + E\eta_1 + ... + E\eta_{n-1}}{n} = a. \tag{28.12}$$

We have:

$$\begin{aligned} E\eta_i = Ef(\xi_i) = \sum_{x\in X} P\{\xi_0 = x\} \cdot E\{f(\xi_i)|\xi_0 = x\} = \sum_{x\in X} \bigl[q_x \cdot \sum_{y\in X} p_{xy}^{(i)} f(y)\bigr] \\ \to \sum_{x\in X} q_x \cdot \sum_{y\in X} p_y \cdot f(y) = a \end{aligned} \tag{28.13}$$

as $i \to \infty$. To prove (28.12), we have to prove that for every positive ε there is a natural n_0 such that the difference of the fraction in (28.12) and a is less than ε in absolute value. According to (28.13), there exists a natural i_0 such that $|E\xi_i - a| < \varepsilon/2$ for $i \geq i_0$. We have for $n \geq i_0$:

$$\begin{aligned} \Bigl|\frac{E\xi_0 + E\xi_1 + ... + E\xi_{n-1}}{n} - a\Bigr| = \Bigl|\frac{\sum_{i=0}^{i_0-1}(E\xi_i - a)}{n} + \frac{\sum_{i=i_0}^{n-1}(E\xi_i - a)}{n}\Bigr| \\ < \frac{|\sum_{i=0}^{i_0-1}(E\xi_i - a)|}{n} + \frac{\varepsilon}{2}. \end{aligned} \tag{28.14}$$

The first term goes to 0 as $n \to \infty$, so the expression (28.14) is less than ε for n not less than some n_0.

Now to the strong law of large numbers.

Theorem 28.3. *Let* ξ_0, ξ_1, ξ_2, ..., ξ_n, ... *be a sequence of random variables with values in a measurable space* $(X, \mathcal{X})$; *and* $f(x)$ *a bounded measurable function on this space.*

If $\alpha^*(\mathcal{F}_{\leq n}, \mathcal{F}_{\geq m}) \leq \beta(m - n)$, $\sum_{k=1}^{\infty} \beta(k) < \infty$, *then strong law of large numbers holds for the sequence of random variables* $\eta_i = f(\xi_i)$, $i = 0, 1, 2, ..., n, ...$:

$$\frac{\eta_0 + \eta_1 + ... + \eta_{n-1}}{n} - \frac{E(\eta_0 + \eta_1 + ... + \eta_{n-1})}{n} \to^{\text{a. s.}} 0 \quad (n \to \infty). \tag{28.15}$$

Proof. Let us introduce the notation: $\zeta_i = \eta_i - E\eta_i$. By Theorem 13.5, it is enough to prove that $\sum_{n=1}^{\infty} E\bigl(\frac{\zeta_0 + ... + \zeta_{n-1}}{n}\bigr)^4 < \infty$. Let us estimate $E(\zeta_0 + ... + \zeta_{n-1})^4$.

We have:

$$E\Bigl(\sum_{i=0}^{n-1}\zeta_i\Bigr)^4 = \sum_{0\le i,\,j,\,k,\,l<n} E(\zeta_i\zeta_j\zeta_k\zeta_l) \le \sum_{0\le i,\,j,\,k,\,l<n} |E(\zeta_i\zeta_j\zeta_k\zeta_l)| \\ \le 24\cdot \sum_{0\le i\le j\le k\le l<n} |E(\zeta_i\zeta_j\zeta_k\zeta_l)| \tag{28.16}$$

(some summands with two or more subscripts coinciding are added when we pass from the sum before the last to the last one).

Let us introduce the notations: $j-i=p$, $k-j=q$, $l-k=r$. Changing summation variables from i, j, k, k to i, p, q, r we can write:

$$\sum_{0\le i\le j\le k\le l<n} |E(\zeta_i\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})| = \sum_{\substack{p,\,q,\,r\ge 0\\ p+q+r<n}}\ \sum_{i=0}^{n-p-q-r-1} |E(\zeta_i\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})|. \tag{28.17}$$

Divide the sum over p, q, r into three: with summands for which p is the largest of p, q, r; with q being the largest of these three differences; and with r being the largest. There will be some summands, say, with $p=q$: let us include them in all three sums:

$$\sum_{\substack{p,\,q,\,r\ge 0\\ p+q+r<n}} \le \sum_{\substack{p,\,q,\,r\ge 0\\ p+q+r<n\\ q,\,r\le p}} + \sum_{\substack{p,\,q,\,r\ge 0\\ p+q+r<n\\ p,\,r\le q}} + \sum_{\substack{p,\,q,\,r\ge 0\\ p+q+r<n\\ p,\,q\le r}}. \tag{28.18}$$

In the first sum we'll estimate the summands $|E(\zeta_i\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})|$ like this: since the random variable ζ_i is not greater than C in absolute value and measurable with respect to $\mathcal{F}_{\le i}$, while $|\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r}| \le C^3$, and $\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r}$ is measurable with respect to $\mathcal{F}_{\ge i+p}$, we have:

$$|E(\zeta_i\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})| \le |E\zeta_i|\cdot|E(\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})| \\ + C^4\cdot\alpha^*(\mathcal{F}_{\le i},\,\mathcal{F}_{\ge i+p}) \le C^4\cdot\beta(p). \tag{28.19}$$

The summands in the third sum in (28.18) are estimated, in the same way, by the same constant $C^4\cdot\beta(r)$. As for the summands in the second sum, we estimate them in a more complicated way:

$$|E(\zeta_i\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})| \\ \le |E(\zeta_i\zeta_{i+p})|\cdot|E(\zeta_{i+p+q}\zeta_{i+p+q+r})| + C^4\cdot\alpha^*(\mathcal{F}_{\le i+p},\,\mathcal{F}_{\ge i+p+q}) \\ \le C^4\cdot[\beta(p)\cdot\beta(r)+\beta(q)] \tag{28.20}$$

(estimates $C^4\cdot\beta(p)$, $C^4\cdot\beta(r)$ are also possible, but they will not yield what we are aiming at).

So we have:

$$\begin{aligned}
\sum_{\substack{p,\,q,\,r\ge 0\\ p+q+r<n}} \sum_{i=0}^{n-p-q-r-1} |E(\zeta_i\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})| &\le \sum_{\substack{p,\,q,\,r\ge 0\\ p+q+r<n}} \sum_{i=0}^{n-1} |E(\zeta_i\zeta_{i+p}\zeta_{i+p+q}\zeta_{i+p+q+r})| \\
&\le C^4\cdot[\sum_{p=0}^{n-1}\sum_{q=0}^{p}\sum_{r=0}^{p} n\cdot\beta(p) + \sum_{q=0}^{n-1}\sum_{p=0}^{q}\sum_{r=0}^{q} n\cdot[\beta(p)\cdot\beta(r)+\beta(q)] \\
&\qquad + \sum_{r=0}^{n-1}\sum_{p=0}^{r}\sum_{q=0}^{r} n\cdot\beta(r)\big] \\
&= n\cdot C^4\cdot[\sum_{p=0}^{n-1}(p+1)^2\cdot\beta(p) + \sum_{q=0}^{n-1}\big(\sum_{p=0}^{q}\beta(p)\big)^2 + \sum_{q=0}^{n-1}(q+1)^2\cdot\beta(q) \\
&\qquad + \sum_{r=0}^{n-1}(r+1)^2\cdot\beta(r)\big] \\
&= n\cdot C^4\cdot\big[3\sum_{p=0}^{n-1}(p+1)^2\cdot\beta(p) + \sum_{q=0}^{n-1}\big(\sum_{p=0}^{q}\beta(p)\big)^2\big] \\
&\le n\cdot C^4\cdot\big[3\sum_{p=0}^{n-1}(p+1)^2\cdot\beta(p) + n\big(\sum_{p=0}^{\infty}\beta(p)\big)^2\big].
\end{aligned} \tag{28.21}$$

Now to the sum $\sum_{n=1}^{\infty} E\big(\frac{\zeta_0+\ldots+\zeta_{n-1}}{n}\big)^4$. This sum is not greater than

$$\begin{aligned}
&24\sum_{n=1}^{\infty} n^{-4}\cdot n\cdot C^4\cdot\big[3\sum_{p=0}^{n-1}(p+1)^2\cdot\beta(p) + n\big(\sum_{p=0}^{\infty}\beta(p)\big)^2\big] \\
&\qquad = 24C^4\cdot\big[3\sum_{p=0}^{\infty}(p+1)^2\cdot\beta(p)\cdot\sum_{n=p+1}^{\infty} n^{-3} + \big(\sum_{p=0}^{\infty}\beta(p)\big)^2\cdot\sum_{n=1}^{\infty} n^{-2}\big].
\end{aligned} \tag{28.22}$$

The series $\sum_{p=0}^{\infty}\beta(p)$ converges, as $\sum_{n=1}^{\infty} n^{-2}$ does, so with the last sum everything is OK. As for $\sum_{n=p+1}^{\infty} n^{-3}$, this sum is not greater than $\int_p^{\infty} x^{-3}\,dx = \frac{1}{2}p^{-2}$ for $p>0$. For $p\ge 1$, we have $p+1\le 2p$, $(p+1)^2\le 4p^2$, and

$$3\sum_{p=0}^{\infty}\big[(p+1)^2\cdot\beta(p)\cdot\sum_{n=p+1}^{\infty} n^{-3}\big] \le 3\beta(0)\cdot\sum_{n=1}^{\infty} n^{-3} + 6\sum_{p=1}^{\infty}\beta(p) < \infty. \tag{28.23}$$

Now the statement of Theorem 28.3 follows from Theorem 13.5.