Lecture 35. About stochastic processes, in general.

In Lecture 25 we had Theorem 25.2: an existence theorem. The way it was proved was by some artificial construction starting with the one-dimensional Lebesgue measure. Unfortunately, it doesn't work in the case of continuous time – so we go by another path, but based ultimately on the same existence theorem for measures (of course, the existence of the Lebesgue measure was proved using the same theorem).

Theorem 35.1. *Let* T *be an arbitrary set; let* ξ_t, $t \in T$, *be a random function* (a family of random variables) *with values in a measurable space* $(X, \mathcal{X})$; *for* $t_1, ..., t_n \in T$, *let* $\mu_{t_1, ..., t_n}$ *be the joint distribution of the random variables* $\xi_{t_1}, ..., \xi_{t_n}$:

$$\mu_{t_1, ..., t_n}(C) = P\{(\xi_{t_1}, ..., \xi_{t_n}) \in C\}, \qquad C \in \mathcal{X}^n. \tag{35.1}$$

Then the distributions $\mu_{t_1, ..., t_n}$ *satisfy the following* consistency conditions:

For every permutation $t_{i_1}, ..., t_{i_n}$ *of the points* $t_1, ..., t_n$ *and for every* $C_1, ..., C_n \in \mathcal{X}$

$$\mu_{t_{i_1}, ..., t_{i_n}}(C_{i_1} \times ... \times C_{i_1}) = \mu_{t_1, ..., t_n}(C_1 \times ... \times C_n); \tag{35.2}$$

for every $t_1, ..., t_n, t_{n+1}$ *and for every* $C_1, ..., C_n \in \mathcal{X}$

$$\mu_{t_1, ..., t_n, t_{n+1}}(C_1 \times ... \times C_n \times X) = \mu_{t_1, ..., t_n}(C_1 \times ... \times C_n). \tag{35.3}$$

Proof. There is practically nothing to prove: the expressions (35.2), (35.3) are both the probabilities of the same event $\{\xi_{t_1} \in C_1, ..., \xi_{t_n} \in C_n\}$, only in one case these inclusions are written in another order, and in the other they are supplemented with the inclusion $\xi_{t_{n+1}} \in X$, which is satisfied for all $\omega \in \Omega$, and can be dropped.

Theorem 35.2 (Kolmogorov's Theorem). *Let* $(X, \mathcal{X}) = (\mathbb{R}^r, \mathcal{B}^r)$; *let* $\mu_{t_1, ..., t_n}$, $t_i \neq t_j$ $(i \neq j)$, $n = 1, 2, 3, ...$, *be a family of probability distributions on* $(X^n, \mathcal{X}^n)$ *satisfying the consistency conditions* (35.2), (35.3).

Then there exists a random function ξ_t, $t \in T$, *whose finite-dimensional joint distributions are* $\mu_{t_1, ..., t_n}$ (i. e., (35.1) is satisfied).

This is a big and pretty complicated theorem.

Of course, it does not make much sense considering finite-dimensional distributions with some $t_i = t_j$: e. g., the "three-dimensional" distribution $\mu_{2,3,2}$ is completely determined by the two-dimensional $\mu_{2,3}$: $\mu_{2,3,2}(C) = \mu_{2,3}\{(x_2, x_3) : (x_2, x_3, x_2) \in C\}$. Nevertheless, the simple Theorem 35.1 remains true for some $t_i = t_j$, so I did not bother to introduce the requirement $t_i \neq t_j$ in it; but Theorem 35.2 is *not* true if we drop this requirement.

Proof of Kolmogorov's Theorem. We take $\Omega = X^T$, the set of all functions x_t, $t \in T$, with values in the space X; this space can be considered as a (possibly infinite-dimensional) coordinate space, x_t being the t-th coordinate of the point $\omega = x_\bullet \in X^T$. We define $\xi_t(\omega) = \xi_t(x_\bullet)$ as the "coordinate" functions: $\xi_t(x_\bullet) = x_t$.

As our fundamental σ-algebra $\mathcal{F}$ we take the *sigma*-algebra in X^T generated by all functions ξ_t, i.e. the smallest σ-algebra in X^T containing all sets of the form $\{x_\bullet : x_t \in C\}$, $C \in \mathcal{X}$. Of course, this definition ensures that the functions ξ_t are measurable with respect to $\mathcal{F}$ (that these functions are *random variables*).

What remains is to define the *probability measure* P on $(\Omega, \mathcal{F})$.

Note that, in contrast with the proof of Theorem 25.2, where the probability measure P was standard (the Lebesgue measure), but the random variables $\xi_0, \xi_1, \xi_2, \dots$ depended on our concrete data q_x, p^n_{xy}, in the present proof the situation is the opposite: the random variables ξ_t are standard, and the probability measure P will depend on the data $\mu_{t_1, \dots, t_n}$.

Let us consider the class of events

$$\mathcal{A} = \bigl\{\{x_\bullet\colon (x_{t_1}, \dots, x_{t_n}) \in C\},\, t_1, \dots, t_n \in T,\, t_i \neq t_j\ (i \neq j),\, C \in \mathcal{X}^n,\, n = 1, 2, 3, \dots\bigr\}. \tag{35.4}$$

On this class of events we define the function P by

$$P\{x_\bullet\colon (x_{t_1}, \dots, x_{t_n}) \in C\} = \mu_{t_1, \dots, t_n}(C). \tag{35.5}$$

However, this being very simple, it is not as simple as that: one and the same event A may be represented in the form $\{x_\bullet\colon (x_{t_1}, \dots, x_{t_n}) \in C\}$ in different ways:

$$A = \{x_\bullet\colon (x_{t_1}, \dots, x_{t_n}) \in C\} = \{x_\bullet\colon (x_{t'_1}, \dots, x_{t'_{n'}}) \in C'\}, \tag{35.6}$$

$C \in \mathcal{X}^n$, $C' \in \mathcal{X}^{n'}$. Will the right-hand sides of the definition (35.5) be the same for these different representations?

The first, and the simplest, case is that of $n' = n$, $t'_1 = t_1, \dots, t'_n = t_n$:

$$A = \{x_\bullet\colon (x_{t_1}, \dots, x_{t_n}) \in C\} = \{x_\bullet\colon (x_{t_1}, \dots, x_{t_n}) \in C'\}. \tag{35.7}$$

If $t_i \neq t_j$ $(i \neq j)$, the set C (as well as C' is the projection of the set $A \subseteq X^T$ onto its coordinates numbers $t_1, \dots, t_n$:

$$C = \{(x_{t_1}, \dots, x_{t_n})\colon x_\bullet \in A\} = C'. \tag{35.8}$$

This is precisely the place in our proof where the condition $t_i \neq t_j$ $(i \neq j)$ is used: without it, it's possible that $C' \neq C$:

$$\{x_\bullet\colon (x_2, x_2) \in [2, 4] \times [3, 6]\} = \{x_\bullet\colon (x_2, x_2) \in [3, \infty) \times (-\infty, 4]\}. \tag{35.9}$$

So in fact the right-hand side of (35.5) depends only on $A \in \mathcal{A}$ and $t_1, \dots, t_n$. Let us denote it as $P_{t_1, \dots, t_n}(A)$.

The next simplest case is that of $n' = n$, and $t'_1 = t_{i_1}, \dots, t'_n = t_{i_n}$, where $i_1, \dots, i_n$ is some permutation of the numbers $1, 2, \dots, n$. We have to prove that

$$P_{t_1, \dots, t_n}(A) = P_{t_{i_1}, \dots, t_{i_n}}(A) \tag{35.10}$$

for every A of the form $A = \{x_\bullet\colon (x_{t_1}, ..., x_{t_n}) \in C\}$ for fixed $t_1, ..., t_n$ and $C \in \mathcal{X}^n$. Both sides are measures; by the consistency condition (35.2) they coincide on the events of the form $A = \{x_\bullet\colon (x_{t_1}, ..., x_{t_n}) \in C_1 \times ... \times C_n\} = \{x_\bullet\colon (x_{t_{i_1}}, ..., x_{t_{i_n}}) \in C_{i_1} \times ... \times C_{i_n}\}$. Such sets form a *semi-algebra* (very easy: e. g., the complement $(C_1 \times ... \times C_n)^c$ is represented as the disjoint union of the products of the form $B_1 \times ... \times B_n$, $B_i = C_i$ or C_i^c, with at least one C_i^c). So by the uniqueness theorem they coincide on the σ-algebra generated by the semi-algebra, i. e., for all events of the form (35.6).

So in fact the set function $P_{t_1, ..., t_n}(A)$ depends only on the *set* of elements t_1, ..., $t_n \in T$, not on their *order*: $P_{t_1, ..., t_n}(A) = P_{\{t_1, ..., t_n\}}(A)$.

Now for the case of $\{t_1, ..., t_n\} \subset \{t'_1, ..., t'_{n'}\}$. We can renumber t'_i so that $t'_1 = t_1$, ..., $t'_n = t_n$, and the rest are just some $t'_{n+1}, ..., t'_{n'} \neq t_i$, $i = 1, ..., n$. In this case it follows from (35.6) that $C' = C \times X \times ... \times X$ (the set C' being the projection of the event A onto the axes number $t_1, ..., t_n, t'_{n+1}, ..., t'_{n'}$). Applying the consistency condition (35.3) several times we get for the events A of the form $A = \{x_\bullet\colon (x_{t_1}, ..., x_{t_n}) \in C_1 \times ... \times C_n\}$ (a semi-algebra):

$$\begin{aligned} P_{\{t_1, ..., t_n, t'_{n+1}, ..., t'_{n'}\}}(A) &= \mu_{t_1, ..., t_n, t'_{n+1}, ..., t'_{n'}}(C_1 \times ... \times C_n \times X \times ... \times X) \\ &= \mu_{t_1, ..., t_n, t'_{n+1}, ..., t'_{n'-1}}(C_1 \times ... \times C_n \times X \times ... \times X) \\ &= ... = \mu_{t_1, ..., t_n}(C_1 \times ... \times C_n) = P_{\{t_1, ..., t_n\}}(A). \end{aligned} \tag{35.11}$$

Finally, for arbitrary $\{t_1, ..., t_n\}$ and $\{t'_1, ..., t_{n'}\}$ we get, by what we have already proved:

$$\mu_{\{t_1, ..., t_n\}}(A) = \mu_{\{t_1, ..., t_n\} \cup \{t'_1, ..., t_{n'}\}}(A) = \mu_{\{t'_1, ..., t_{n'}\}}(A). \tag{35.12}$$

So now we have proved that the formula (35.5) indeed defines a set function on the class $\mathcal{A}$. What remains to prove is that the class $\mathcal{A}$ is an algebra (this is easy) and that the set function P on this class is countably additive.

I am skipping the algebra proof. Let us prove that P is *finitely* additive on $\mathcal{A}$.

We have to prove that for disjoint events (sets in X^T) $A_1 = \{(\xi_{t_1}, ..., \xi_{t_n}) \in C_1\}$, $A_2 = \{(\xi_{t'_1}, ..., \xi_{t'_{n'}}) \in C_2\}$ we have $P(A_1 \cup A_2) = P(A_1) \cup P(A_2)$.

If $n' = n$, $t'_1 = t_1$, ..., $t'_n = t_n$, this is obvious, because the projections C_1 and C_2 are disjoint, $P(A_1 \cup A_2) = \mu_{t_1, ..., t_n}(C_1 \cup C_2)$, and $\mu_{t_1, ..., t_n}$ is a measure. The same, of course, if $n' = n$, and t'_i go in some permuted order. In the general case we take the union $\{s_1, ..., s_m\} = \{t_1, ..., t_n\} \cup \{t'_1, ..., t'_{n'}\}$ and use the additivity of the measure $\mu_{s_1, ..., s_m}$.

I cannot remember whether I gave you the following little measure-theory theorem:

Theorem 35.3. *Let m be a finite finitely additive set function on an algebra A of subsets of X. This set function is* countably *additive if and only if it is "continuous at zero"*:

$$A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ..., \quad \bigcap_{i=1}^{\infty} A_i = \emptyset \quad \Rightarrow \quad \lim_{n\to\infty} m(A_i) = 0. \tag{35.13}$$

Anyway, I am skipping the proof of this purely measure-theoretic theorem.

So we have to prove that for events $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ of the form (35.6) with an empty intersection we have $\lim_{n\to\infty} P(A_n) = 0$. Or, which is the same: if $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$, and $\lim_{n\to\infty} P(A_n) > 0$, then the intersection of these events is not empty.

For what follows we need a theorem belonging to more difficult areas of the field of measure theory (I am giving it without proof):

Theorem 35.4. *Let m be a finite measure on $(\mathbb{R}^m, \mathcal{B}^m)$. Then for every $C \in \mathcal{B}^m$ and every positive ε there exists a compact set $K \subseteq C$ such that $m(C \setminus K) < \varepsilon$.*

Now we go to proving Kolmogorov's Theorem. Of course there is nothing to prove if the parameter set T is finite, having n elements: we just take $P = \mu_{t_1, \ldots, t_n}$. For an infinite T we assume that there is a non-increasing sequence of sets $A_i \subseteq X^T$ and a sequence $t_1, t_2, \ldots, t_n, \ldots$ such that $A_n = \{x_\bullet\colon (x_{t_1}, \ldots, x_{t_n}) \in C_n$, $C_n \in \mathcal{X}^n$ (if more than one element of T is added in passing from A_n to A_{n+1}, we repeat the same A_n several times, representing it instead of $A_n = \{x_\bullet\colon (x_{t_1}, \ldots, x_{t_n}) \in C_n$ as $A_n = \{x_\bullet\colon (x_{t_1}, \ldots, x_{t_n}, x_{t_{n+1}}) \in C_n \times X\}$). We suppose that $\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \mu_{t_1, \ldots, t_n}(C_n) > 0$. Let us choose a positive $\varepsilon < \lim_{n\to\infty} \mu_{t_1, \ldots, t_n}(C_n)$.

We'll finish the proof in the next lecture.