Lecture 36. End of the proof of Kolmogorov's Theorem. Reformulations.

Let us finish the proof of Kolmogorov's Theorem.

For every natural n we choose, by Theorem 35.3, a compact set $K_n \subseteq C_n$ such that $\mu_{t_1,\,...,\,t_n}(C_n \setminus K_n) < \varepsilon/2^n$. These sets are clearly non-empty (because $\mu_{t_1,\,...,\,t_n}(K_n) = \mu_{t_1,\,...,\,t_n}(K_n) - \mu_{t_1,\,...,\,t_n}(C_n \setminus K_n) > 0$). We want to use the well-known fact that the intersection of an infinite non-increasing sequence of non-empty compact sets is non-empty.

Unfortunately, the sequence of sets K_n is absolutely *not* non-increasing: to begin with, all these sets are in spaces of different dimensions: $K_n \subset X^n$. But the sequence of sets $B_n = \{x_\bullet\colon (x_{t_1}, ..., x_{t_n}) \in K_n\}$, which are in the same space X^T, is not necessarily non-decreasing either (in contrast with the sequence $A_n = \{x_\bullet\colon (x_{t_1}, ..., x_{t_n}) \in C_n\}$). To make this right, we take the sequence of events (sets in X^T)

$$B_n' = B_1 \cap B_2 \cap ... \cap B_n. \tag{36.1}$$

We have, obviously, $B_1' = B_1 \supseteq B_2' \supseteq ... \supseteq B_n' \supseteq ...$, and all these (infinite-dimensional) sets are non-empty:

$$\begin{aligned} P(B_n') = P(B_1 \cap B_2 \cap ... \cap B_n) &\ge P(A_1 \cap A_1 \cap ... \cap A_n) - \sum_{i=1}^n P(A_i \setminus B_i) \\ &= P(A_n) - \sum_{i=1}^n \mu_{t_1,\,...,\,t_n}(C_i \setminus K_i) > \varepsilon - \sum_{i=1}^n \varepsilon/2^i = \varepsilon/2^n > 0. \end{aligned} \tag{36.2}$$

Every set B_n' is represented as $B_n' = \{x_\bullet\colon (x_{t_1}, ..., x_{t_n}) \in K_n'\}$, where

$$K_n' = K_n \cap (K_{n-1} \times X) \cap (K_{n-2} \times X^2) \cap ... \cap (K_1 \times X^{n-1}). \tag{36.3}$$

The set K_n' is a compact set as the intersection of one compact set K_n and $n-1$ closed ones, $K_i \times X^{n-i}$.

Now for every n we consider the sequence $K''_{n,\,n} = K_n'$, $K''_{n,\,n+1}$, $K''_{n,\,n+2}$, ..., where the set $K''_{n,\,i}$, $i \ge n$, is the projection of the infinite-dimensional set B_n' onto the coordinates number t_1, ..., t_n:

$$K''_{n,\,i} = \{(x_{t_1}, ..., x_{t_n})\colon x_\bullet \in B_i'\}. \tag{36.4}$$

Clearly, the sets $K''_{n,\,i}$, $i \ge n$, form a non-increasing sequence.

We can also represent $K''_{n,\,i}$ as the projection of the i-dimensional set onto the first n coordinates:

$$K''_{n,\,i} = \{(x_{t_1}, ..., x_{t_n})\colon (x_{t_1}, ..., x_{t_n}, x_{t_{n+1}}, ..., x_{t_i}) \in K_i'\}. \tag{36.5}$$

Clearly all these sets are compact as projections of compact sets.

Now, the one-dimensional set $\bigcap_{i=1}^{\infty} K''_{1,\,i}$ is non-empty. Choose a point $x^*_{t_1}$ in this intersection.

Since this point belongs to all projections of the sets B'_i onto the one-dimensional space, for every $i > 1$ there exists a point in K'_i with the first coordinate $x^*_{t_1}$. This means that the intersection $K'_i \cap \{(x_{t_1}, ..., x_{t_n}) : \; x_{t_1} = x^*_{t_1}\}$ is a non-empty i-dimensional set. This set is compact as the intersection of a compact set K'_i and an $(i-1)$-dimensional hyperplane, which is a closed set. Also non-empty and compact are their 2-dimensional projections $K''_{2,\,i} \cap \{(x^*_{t_1}, x_{t_2}) \colon x_{t_{i_2}} \in X\}$.

Now choose a point $(x^*_{t_1}, x^*_{t_2})$ belonging to all sets $K''_{2,\,i}$, $i \geq 2$; then a point $(x^*_{t_1}, x^*_{t_2}, x^*_{t_3})$ belonging to all projections $K''_{3,\,i}$; etc. We obtain an infinite sequence $x^*_{t_1}$, $x^*_{t_2}$, $x^*_{t_3}$, ..., $x^*_{t_n}$, ... Then we take a function $x^*_\bullet \in X^T$ taking the prescribed values at the points t_1, t_2, ..., t_n, ...; at the points $t \neq t_n$ (if there are left any) we define this function arbitrarily. The function $x^*_\bullet$ belongs to all sets B'_n.

Indeed, the set B'_n consists of all functions $x_\bullet$ for which the n-dimensional point $(x_{t_1}, ..., x_{t_n}) \in K'_n$, and $(x^*_{t_1}, ..., x^*_{t_n})$ does belong to the set $K''_{n,\,n} = K'_n$.

So the set $\bigcap_{n=1}^{\infty} B'_n$ is non-empty, and with it also a larger set $\bigcap_{n=1}^{\infty} B_n$.

The theorem is proved.

The above proof is that given originally by Kolmogorov, and can be read in his book *Foundations of the Theory of Probability*, New York, Chelsea Pub. Co., 1950.

Let us look at other formulations of the same theorem; we'll need some of them.

First of all, the theorem remains true for the case of the space $(X, \mathcal{X})$ being not a Euclidean space $\mathbb{R}^r$ with its Borel σ-algebra $\mathcal{B}^r$, but for X being a Borel subset of $\mathcal{R}^r$, and $\mathcal{X}$ the class of its Borel subsets: $\mathcal{X} = \mathcal{B}_X = \{A : \; A \in \mathcal{B}^r, \, A \subseteq X\}$. I did not formulate Kolmogorov's Theorem with an arbitrary Borel $X \subseteq \mathbb{R}^r$ because I wanted to keep the formulation as simple as possible. The only place where what $(X, \mathcal{X})$ is matters is Theorem 35.4; and here we can extend an arbitrary measure m on $\mathcal{B}_X$ to the whole $\mathcal{B}^m$ by defining $\tilde{m}(C) = m(C \cap X)$.

In particular, Kolmogorov's Theorem is true for the case of a countable (finite or infinite) X and $\mathcal{X}$ being the σ-algebra of all its subsets (we can put a countable set isomorphically into a Euclidean space); i.e. for discrete distributions. For this case, we can reformulate the consistency conditions (35.2), (35.3) in terms of probability mass functions $p_{t_1,\,...,\,t_n}(x_1, ..., x_n) = \mu_{t_1,\,...,\,t_n}\{(x_1, ..., x_n)\}$:

$$p_{t_{i_1},\,...,\,t_{i_n}}(x_{i_1}, ..., x_{i_n}) = p_{t_1,\,...,\,t_n}(x_1, ..., x_n), \tag{36.6}$$

$$p_{t_1,\,...,\,t_n}(x_1, ..., x_n) = \sum_{x_{n+1}} p_{t_1,\,...,\,t_n,\,t_{n+1}}(x_1, ..., x_n, x_{n+1}). \tag{36.7}$$

We can also rewrite the consistency conditions for distributions with densities $p_{t_1,\,...,\,t_n}(x_1, ..., x_n)$: the first condition has the same form (36.6) as in the discrete case (but the meaning of the function $p_{t_1,\,...,\,t_n}(x_1, ..., x_n)$ in it is quite different); and instead of (36.7) we get

$$p_{t_1,\,...,\,t_n}(x_1, ..., x_n) = \int_X p_{t_1,\,...,\,t_n,\,t_{n+1}}(x_1, ..., x_n, x_{n+1}) \, dx_{n+1}. \tag{36.8}$$

Then, if we know the finite-dimensional distribution $\mu_{2,3,5}$, we know also the distributions $\mu_{2,5,3}$, $\mu_{3,2,5}$, etc. for all permutations; so why not restrict ourselves to *one* of these distributions, and obtain the remaining $n!-1$ automatically? But which of the orders t_1, ..., t_n to choose as the basic one? If there is *order* in the parameter set T, it's easy: we choose the order for which $t_1 < t_2 < ... < t_n$. So for the case of $T \subseteq \mathbb{R}^1$ we can reformulate our Theorems 35.1, 35.2 for a smaller collection of distributions $\mu_{t_1,...,t_n}$ with $t_1 < t_2 < ... < t_n$ (or $t_1 \le t_2 \le ... \le t_n$ in Theorem 35.1). Of course we don't need any consistency condition of the type (35.2): we cannot permute the points $t_1 < t_2 < ... < t_n$ keeping them in the same increasing order. But instead of the consistency condition (35.3) we'll have *several* conditions.

Indeed, in the condition (36.3) we added one new time moment t_{n+1} – putting it at the end of the sequence t_1, ..., t_n. But the moment we add is not necessarily greater than all previous ones; so instead of one condition (35.3) we'll have $n+1$ conditions: for the case that the new time moment being smaller than all previous ones; for the case that the new time moment is the second smallest; ...; and finally for the case that it is larger than all others:

$$\mu_{t_1,t_2,...,t_{n+1}}(X \times C_2 \times ... \times C_{n+1}) = \mu_{t_2,...,t_{n+1}}(C_2 \times ... \times C_{n+1}), \tag{36.9$_1$}$$

. . . . .

$$\begin{aligned}\mu_{t_1,...,t_{i-1},t_i,t_{i+1},...,t_n}&(C_1 \times ... \times C_{i-1} \times X \times C_{i+1} \times ... \times C_{n+1}) \\ &= \mu_{t_1,...,t_{i-1},t_{i+1},...,t_n}(C_1 \times ... \times C_{i-1} \times C_{i+1} \times ... \times C_{n+1}),\end{aligned} \tag{36.9$_i$}$$

. . . . .

$$\mu_{t_1,...,t_n,t_{n+1}}(C_1 \times ... \times C_n \times X) = \mu_{t_1,...,t_n}(C_1 \times ... \times C_n), \tag{36.9$_{n+1}$}$$