

Lecture 37. Application to Markov processes. The Wiener process.

Theorem 37.1. *Let ν be a probability measure on the space (X, \mathcal{X}) , and let $P(s, x, t, C)$ be a function of the times $s, t, 0 \leq s \leq t$, of $x \in X$, and of a set $C \in \mathcal{X}$, such that for fixed s, t , and x it is a probability measure as a function of C ; for fixed s, t , and C it is measurable as a function of x ; $P(t, x, t, C) = \delta_x(C)$; and this function satisfies the Chapman – Kolmogorov equation.*

Then the finite-dimensional distributions μ_{t_1, \dots, t_n} defined, for $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$, by

$$\begin{aligned} \mu_{t_1, \dots, t_n}(C) = & \int_X \nu(dx_0) \int_X P(t_0, x_0, t_1, dx_1) \int_X P(t_1, x_1, t_2, dx_2) \int_X \dots \\ & \dots \int_X P(t_{n-1}, x_{n-1}, t_n, dx_n) I_C(x_1, \dots, x_n) \end{aligned} \tag{37.1}$$

satisfy the consistency conditions.

The **proof** is very simple: it could seem that the conditions imposed on $P(s, x, t, C)$ were designed specially so that the proof is easy. The condition (36.9_{n+1}) is satisfied because $P(t_n, x_n, t_{n+1}, X) = 1$, and conditions (9_i), $1 \leq i \leq n$, because of the Chapman – Kolmogorov equality $\int_X P(t_{i-1}, x_{i-1}, t_i, dx_i) P(t_i, x_i, t_{i+1}, \bullet) = P(t_{i-1}, x_{i-1}, t_{i+1}, \bullet)$.

So if the space (X, \mathcal{X}) is not extremely ugly (e.g. a non-Borel set with I don't know what as the σ -algebra in it), there exists a stochastic process with the prescribed finite-dimensional distributions, and by Theorem 34.1 it is a Markov process with initial distribution ν and transition function $P(\cdot, \cdot, \cdot, \cdot)$. In particular, there is a Markov process corresponding to the transition function you invented solving Problem **54**. *Can you say anything about this Markov process of yours, apart from its finite-dimensional distributions?*

Another example:

Example 37.1. Take $(X, \mathcal{X}) = (\mathbb{R}^1, \mathcal{B}^1)$. The transition density

$$p(s, x, t, y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)} \tag{37.2}$$

(as a function of y , it is the density of the normal distribution with parameters $(x, t-s)$) satisfies the Chapman – Kolmogorov equation:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)} \cdot \frac{1}{\sqrt{2\pi(u-t)}} e^{-(z-y)^2/2(u-t)} dy \\ = \frac{1}{\sqrt{2\pi(u-s)}} e^{-(z-x)^2/2(u-s)}. \end{aligned} \tag{37.3}$$

One of the ways to check it is to collect all terms with y^2 , with y , and with no y in the quadratic function being the exponent in the integrand, and use the fact that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi b}} e^{-(u-a)^2/2b} du = 1$.

Another way: For fixed x , the integrand in (37.3) is, as a function of y, z , the density of the two-dimensional normal distribution with parameters $\left(\begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} t-s & t-s \\ t-s & u-s \end{pmatrix}\right)$ (check it!); the integral with respect to y , as a function of z , represents the density of the second coordinate: that of the normal distribution with parameters $(x, u-s)$.

A third way: the left-hand side integral in (37.3) is the convolution formula for the probability density of the sum of two independent one-dimensional random variables ξ and η , where ξ has the normal distribution with parameters $(x, t-s)$, and the second, η , with parameters $(0, u-t)$. The characteristic function $f_{\xi+\eta}(v) = Ee^{v\cdot(\xi+\eta)}$ of the sum is equal to the product of their characteristic functions: $f_{\xi+\eta}(v) = f_{\xi}(v) \cdot f_{\eta}(v) = e^{ixv-(t-s)\cdot v^2/2} \cdot e^{-(u-t)\cdot v^2/2} = e^{ixv-(u-s)\cdot v^2/2}$: the characteristic function corresponding to the normal density in the right-hand side of (37.3) (you see, I am running out of letters).

Problem [A] Check that the density $p(s, x, t, y) = \frac{\pi^{-1}(t-s)}{(t-s)^2 + (y-x)^2}$ satisfies the Chapman–Kolmogorov equation.

A stochastic process is a function $\xi_t(\omega)$ of two arguments: the “time” $t \in T \subseteq \mathbb{R}^1$, and the sample point $\omega \in \Omega$.

If we fix $t \in T$, the function $\xi_t(\bullet)$ of the second argument is a random variable.

If we fix ω , $\xi_{\bullet}(\omega)$ is just a function of “time”. We call this function a *trajectory* (a *sample function*) of our stochastic process.

A stochastic process $\xi_t, t \in [0, \infty)$, with values in $(\mathbb{R}^1, \mathcal{B}^1)$ is called a *Wiener process* if its finite-dimensional distributions μ_{t_1, \dots, t_n} are given by formula (37.1) with

$$P(s, x, t, C) = \int_C \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)} dy, \quad 0 \leq s < t, \quad (37.4)$$

$$P(t, x, t, C) = \delta_x(C);$$

and all its trajectories are continuous: $\xi_t(\omega)$ is continuous in t for every $\omega \in \Omega$.

We have proved that it is possible to satisfy the first part of the definition – that about the finite-dimensional distributions; but we know nothing about the second part – about all trajectories being continuous. The credit for this belongs to N.Wiener.

The Wiener process is the mathematical model for the physical phenomenon of Brownian motion: the process of motion of a light particle in a fluid under the influence of chaotic impulses from molecules hitting it. That is, the Brownian motion is three-dimensional (or two-dimensional if we want to model what we see under the microscope), while we consider the one-dimensional Wiener process; more accurate would be saying that the Wiener process is the mathematical model of one coordinate of the physical Brownian motion.

The stochastic process $\xi_t, t \in [0, \infty)$, having the given finite-dimensional distributions (no matter what these distributions are) constructed in the proof of Kolmogorov’s Theorem

(Theorem 35.2) clearly doesn't have all trajectories continuous: the trajectories $\xi_t(\omega) = \xi_t(x_\bullet) = x_t$ are *all* functions belonging to $\mathbb{R}^{[0, \infty)}$ (all real-valued functions on the interval $[0, \infty)$), and we know that there *are* discontinuous functions. We cannot state either that *almost all* functions $\xi_t(x_\bullet)$ are continuous: it can be proved easily that $P\{x_\bullet : x_t \text{ is continuous}\}$ makes no sense: the set $\{x_\bullet : x_t \text{ is continuous}\}$ does not belong to the σ -algebra $\mathcal{B}^{[0, \infty)}$.

We may hope that there is, on the same probability space $(\mathbb{R}^{[0, \infty)}, \mathcal{B}^{[0, \infty)}, P)$, another stochastic process $\tilde{\xi}_t, t \in [0, \infty)$, with the same finite-dimensional distributions, but with continuous trajectories.

Theorem 37.2. *If $\xi_t, t \in T$, and $\tilde{\xi}_t, t \in T$, are two random functions, and $P\{\tilde{\xi}_t \neq \xi_t\} = 0$ for every $t \in T$, then the random functions $\tilde{\xi}_t$ and ξ_t have the same finite-dimensional distributions.*

Proof. We have to prove that for every natural n , for every $t_1, \dots, t_n \in T$, and for every $C \in \mathcal{X}^n$

$$P\{(\tilde{\xi}_{t_1}, \dots, \tilde{\xi}_{t_n}) \in C\} = P\{(\xi_{t_1}, \dots, \xi_{t_n}) \in C\}. \quad (37.5)$$

The symmetric difference of the events $\{(\tilde{\xi}_{t_1}, \dots, \tilde{\xi}_{t_n}) \in C\}$ and $\{(\xi_{t_1}, \dots, \xi_{t_n}) \in C\}$ is clearly a subset of the event $\{\tilde{\xi}_{t_1} \neq \xi_{t_1}\} \cup \dots \cup \{\tilde{\xi}_{t_n} \neq \xi_{t_n}\}$, and the probability of this union is not greater than $\sum_{i=1}^n P\{\tilde{\xi}_{t_i} \neq \xi_{t_i}\} = 0$.

In the next lecture we'll have a bigger theorem that will help us to prove the existence of a Wiener process.