Lecture 38. Another Kolmogorov's theorem. The Wiener process.

Now to a big theorem:

Theorem 38.1 (also proved by Kolmogorov). *Let* ξ_t, $0 \le t \le T$, *be a* (real-valued) *stochastic process such that for some* $\alpha > 0$, *some* $\beta > 0$, *and some* C *for all* $s, t \in [0, T]$, *we have*:

$$E|\xi_t - \xi_s|^\alpha \le C \cdot |t - s|^{1+\beta}. \tag{38.1}$$

Then there exists a stochastic process $\tilde{\xi}_t$ *whose all trajectories are continuous, and* $P\{\tilde{\xi}_t \ne \xi_t\} = 0$ *for every* $t \in [0, T]$.

Before I prove this theorem, let me tell you that without great effort from this theorem for every finite interval $[0, T]$ the corresponding theorem for the infinite time interval is deduced; and that I will be proving the theorem for the interval $[0, 1]$ (this makes no difference: just a little longer notations).

Proof. Let T_0 be the subset of the interval $[0, 1]$ consisting of all binary-rational numbers: $T_0 = \{\frac{k}{2^n} :\ k = 0, 1, 2, 3, ..., 2^n,\ n = 1, 2, 3, ...\}$. Let

$$\eta_n = \sum_{k=1}^{2^n} |\xi_{k/2^n} - \xi_{(k-1)/2^n}|^\alpha. \tag{38.2}$$

By (38.1), we have: $E\eta_n \le 2^n \cdot C \cdot (2^{-n})^{1+\beta} = C \cdot 2^{-n\beta}$. Let us choose some positive $\gamma < \beta$. Then

$$E \sum_{n=1}^{\infty} \eta_n \cdot 2^{\gamma n} = \sum_{n=1}^{\infty} E\eta_n \cdot 2^{\gamma n} \le C \cdot \sum_{n=1}^{\infty} 2^{-n(\beta-\gamma)} = \frac{C \cdot 2^{-(\beta-\gamma)}}{1 - 2^{-(\beta-\gamma)}} < \infty, \tag{38.3}$$

and the random variable $\mathrm{H} = \sum_{n=1}^{\infty} \eta_n \cdot 2^{n\gamma}$ is finite almost surely (the letter "capital eta" is written exactly like H).

It is clear that for every natural n and every k between 1 and 2^n

$$|\xi_{k/2^n} - \xi_{(k-1)/2^n}| = (|\xi_{k/2^n} - \xi_{(k-1)/2^n}|^\alpha)^{1/\alpha} \le \Big(\sum_{k=1}^{2^n} |\xi_{k/2^n} - \xi_{(k-1)/2^n}|^\alpha\Big)^{1/\alpha} = \eta_n^{1/\alpha}. \tag{38.4}$$

Also it is clear that

$$\eta_n = 2^{-n\gamma} \cdot (\eta_n \cdot 2^{n\gamma}) \le 2^{-n\gamma} \cdot \sum_{n=1}^{\infty} \eta_n \cdot 2^{n\gamma} = 2^{-n\gamma} \cdot \mathrm{H}, \tag{38.5}$$

so that

$$|\xi_{k/2^n} - \xi_{(k-1)/2^n}| \le \mathrm{H}^{1/\alpha} \cdot 2^{-n\gamma/\alpha}. \tag{38.6}$$

Let us prove that if $\mathrm{H} < \infty$, the function $\xi_t(\omega)$, $t \in T_0$, is *uniformly continuous.* It's enough to prove that there exists a function $\varepsilon(h)$, $h > 0$, such that $\lim_{h \to 0+} \varepsilon(h) = 0$, and

$$|\xi_t(\omega) - \xi_s(\omega)| \le \varepsilon(|t-s|) \qquad \text{for all } s,\, t \in T_0. \tag{38.7}$$

Of course it's enough to prove (38.7) only for $t > s$. Let $s < t$ be some binary-rational points between 0 and 1, and let n_0 be such that $\dfrac{1}{2^{n_0}} < t - s \le \dfrac{1}{2^{n_0 - 1}}$. Then between the points s and t lie two or three points of the form $\dfrac{k}{2^{n_0}}$ (make a picture; make it pretty large); and between these two points lie at most two intervals of the form $[(k-1)/2^{n_0},\, k/2^{n_0}]$. We have by (38.6) $|\xi_{k/2^{n_0}} - \xi_{(k-1)/2^{n_0}}| \le \mathrm{H}^{1/\alpha} \cdot 2^{-n_0 \gamma/\alpha}$.

Between the right end of the rightmost of these intervals of length $1/2^{n_0}$ and the point t lies at most one interval of the form $[(k-1)/2^{n_0+1},\, k/2^{n_0+1}]$ (k being different from what was denoted with k right the previous moment), and the increment of the function $\xi_\bullet$ on this interval is not greater in absolute value than $\mathrm{H}^{1/\alpha} \cdot 2^{-(n_0+1)\gamma/\alpha}$; between its end and the point t at most one interval of the form $[(k-1)/2^{n_0+2},\, k/2^{n_0+2}]$, with increment at most $\mathrm{H}^{1/\alpha} \cdot 2^{-(n_0+2)\gamma/\alpha}$; etc. Finally we come to t. The same way we go to the left to the point s. Finally we get that the difference $\xi_t - \xi_s$ is represented as a finite sum of increments of the function $\xi_\bullet$ on the intervals of the form $[(k-1)/2^n,\, k/2^n]$, $n \ge n_0$, and there are no more than two intervals for every $n \ge n_0$ (in fact, zero intervals for sufficiently large n). The absolute value of the sum is not greater than the sum of absolute values, so

$$|\xi_t - \xi_s| < 2 \sum_{n=n_0}^{\infty} \mathrm{H}^{1/\alpha} \cdot 2^{-n\gamma/\alpha} = \frac{2\mathrm{H}^{1/\alpha} \cdot 2^{-n_0\gamma/\alpha}}{1 - 2^{-\gamma/\alpha}}. \tag{38.8}$$

Now we remember our choice of n_0: it was such that $2^{-n_0} < t - s$; so we have $2^{-n_0\gamma/\alpha} < (t-s)^{\gamma/\alpha}$, and

$$|\xi_t - \xi_s| < \frac{2\mathrm{H}^{1/\alpha}}{1 - 2^{-\gamma/\alpha}} \cdot (t-s)^{\gamma/\alpha}. \tag{38.9}$$

So inequality (38.7) is satisfied with $\varepsilon(h) = \dfrac{2\mathrm{H}^{1/\alpha}}{1 - 2^{-\gamma/\alpha}} \cdot h^{\gamma/\alpha}$, the uniform continuity is proved.

Now we know that a uniformly continuous function defined on a set T_0 has a limit at every point that is a limit point of the set T_0; in our case these limit points are all points of the interval $[0, 1]$. So we can define for all $t \in [0, 1]$

$$\tilde{\xi}_t(\omega) = \begin{cases} \lim\limits_{T_0 \ni t' \to t} \xi_{t'}(\omega) & \text{if } \mathrm{H}(\omega) < \infty, \\ 0 & \text{if } \mathrm{H}(\omega) = \infty \end{cases} \tag{38.10}$$

(the second possibility takes place only with probability 0; we could have taken, instead of the identical 0, any other continuous function, e. g., $\sin t$, or $\cos(\xi_{2/3} \cdot t)$).

For every $t \in [0, 1]$, the function $\tilde{\xi}_t(\omega)$ is a random variable (i. e. measurable with respect to $\mathcal{F}$): the set $\{\omega\colon\ \mathrm{H}(\omega) < \infty\} \in \mathcal{F}$ (that is, is an event), and $\lim_{T_0 \ni t' \to t} \xi_{t'}(\omega)$

can be represented as the limit of a sequence $\lim_{n\to\infty} \xi_{t_n}$, where t_n are binary-rational points such that $t_n \to t$.

So we do have a stochastic process.

What remains to be proved is that for every $t \in [0, 1]$ we have $P\{\tilde{\xi}_t = \xi_t\} = 1$.

Let t_n, $n = 1, 2, 3, \ldots$, be a sequence of binary-rational numbers converging to the number t. Then almost surely (on the set $\{\omega : \mathrm{H}(\omega) < \infty\}$) $\xi_{t_n} \to \tilde{\xi}_t$ as $n \to \infty$. On the other hand, $\xi_{t_n} \to_P \xi_t$ $(n \to \infty)$, because $P\{|\xi_{t_n} - \xi_t| \geq \varepsilon\} \leq \dfrac{E|\xi_{t_n} - \xi_t|^\alpha}{\varepsilon^\alpha} \leq \dfrac{E|t_n - t|^{1+\beta}}{\varepsilon^\alpha} \to 0$ $(n \to \infty)$. It follows from $\xi_{t_n} \to^{\text{a.s.}} \tilde{\xi}_t$ and $\xi_{t_n} \to_P \xi_t$ that $\tilde{\xi}_t = \xi_t$ almost surely.

The theorem is proved.

Now we apply this to the process with the finite-dimensional distribution given by formulas (37.1), (37.4). I am giving this as a series of problems.

$\boxed{\mathbf{B}}$ For $0 \leq s < t$, find the conditional distribution of the difference $\xi_t - \xi_s$ with respect to the σ-algebra $\mathcal{F}_{[0,\, s]}$ generated by the random variables ξ_v, $0 \leq v \leq s$.

$\boxed{\mathbf{C}}$ Prove that $\xi_t - \xi_s$ has the normal distribution with parameters $(0, t - s)$.

$\boxed{\mathbf{D}}$ Prove that the random variables $\xi_s - \xi_0$, $\xi_t - \xi_s$ are independent.

The same way as we solve Problem $\boxed{\mathbf{D}}$, only with far more to write, we can prove that for $t_0 < t_1 < \ldots < t_n$ the increments $\xi_{t_1} - \xi_{t_0}, \xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent.

$\boxed{\mathbf{E}}$ Prove that $E|\xi_t - \xi_s|^4 = 3(t - s)^2$.

So we see that the conditions of Theorem 38.1 are satisfied with $\alpha = 4$, $\beta = 1$; and a Wiener process does exist.