Problems.

[1] Specialists in probability theory have endeavored to apply it to number theory; e. g. Chebyshev.

For some subsets A of the set N of natural numbers what may be called *asymptotic density* exists:

$$\nu(A) = \lim_{n\to\infty} \frac{\text{number of elements in } A \cap \{1, 2, ..., n\}}{n}. \qquad (*)$$

For example, for the set D_3 of natural numbers divisible by 3 we have $\nu(D_3) = 1/3$; for the set Pr of all prime numbers $\nu(Pr) = 0$ (even if it is not so easy to prove as in the case of D_3).

Let $\mathcal{N}$ be the class of sets for which the limit $(*)$ exists.

Is the class $\mathcal{N}$ a σ-algebra in N?

Is it an algebra?

[2] Is the set function ν of the previous problem finitely additive (i. e., does, for disjoint A_1, ..., $A_n \in \mathcal{N}$ such that $\bigcup_{i=1}^n A_i \in \mathcal{N}$, the equality $\nu\big(\bigcup_{i=1}^n A_i\big) = \sum_{i=1}^n \nu(A_i)$ necessarily hold?)?

[3] Is the set function ν of problem **[1]** countably additive?

[4] Can you produce an example of a probability space and a sequence of events A_i in it, such that $\sum_{i=1}^\infty P(A_i) = \infty$, $P\{\text{infinitely many of } A_i \text{ occur}\} = 1$?

[5] Can you produce an example of a probability space and a sequence of events A_i in it, such that $\sum_{i=1}^\infty P(A_i) = \infty$, $P\{\text{infinitely many of } A_i \text{ occur}\} \in (0,\ 1)$?

[6] Can you produce an example of a probability space and a sequence of events A_i in it, such that $\sum_{i=1}^\infty P(A_i) = \infty$, $P\{\text{infinitely many of } A_i \text{ occur}\} = 0$?

The deadline for Problems **[1]**–**[6]** is Sep. 17.

[7] Let $\mathcal{I}^2$ be the class of all rectangles $\{(x,\ y)\colon\ a < x \le b,\ c < y \le d\}$, finite or infinite, in the plane $\mathbb{R}^2$ (if, say, $b = \infty$, the inequality $x \le \infty$ means the same as $x < \infty$: no real number x is *equal* to ∞); let $\mathcal{O}$ be the class of all open sets in $\mathbb{R}^2$.

Prove that $\sigma(\mathcal{I}^2) = \sigma(\mathcal{O})$.

[8] Let X be a Borel set in $\mathbb{R}^n$. Prove that the class of all Borel subsets of X is a σ-algebra in X.

Prove that this σ-algebra is the same as that generated by all subsets of X that are open in X.

The deadline for Problems **[7]**–**[8]** is Sep. 19.

9 (the deadline Sep. 22). Let μ be the one-dimensional normal distribution with parameters (a, b) ($a \in (-\infty, \infty)$, $b \in (0, \infty)$); i.e., a continuous distribution on $\mathbb{R}^1$ with density

$$p(x) = \frac{1}{\sqrt{2\pi b}}\, e^{-(x-a)^2/2b} \qquad (*)$$

(this function is positive, and it can be checked that its integral over the whole line is equal to 1).

What is the result of putting this distribution through the filter with $f(x) = e^{cx}$?

10 Prove that if ξ and η are two random variables (on the same sample space, and taking values in $\mathbb{R}^1$), then $\xi + \eta$ is also a random variable.

11 Prove or disprove: Let $F(t)$, $-\infty < t < \infty$, be a right-continuous nondecreasing function, $F(-\infty) = 0$, $F(+\infty) = 1$. If F has a finite number of jumps at the points $x_m < x_{m+1} < ... < x_n$, and is constant on the intervals $(-\infty, x_m)$, (x_m, x_{m+1}), ..., (x_{n-1}, x_n), (x_n, ∞), then this function is a distribution function of a discrete random variable.

12 Prove or disprove the statement of the previous problem with the finite sequence of jump points replaced by a countable one that is infinite on one side: $x_m < x_{m+1} < ... < x_k < x_{k+1} < ...$, or $... < x_k < x_{k+1} < ... < x_n$; or on both: $... < x_{-1} < x_0 < x_1 < x_2 < ...$.

13 Prove or disprove: Let $F(x)$, $-\infty < x < \infty$, be a right-continuous nondecreasing function, $F(-\infty) = 0$, $F(+\infty) = 1$. If

$$\sum_x [F(x) - F(x^-)] = 1,$$

then the function F is a distribution function of a discrete random variable.

(The sum has, in appearance, an uncountable number of summands, but in fact all of them except a countable number are equal to 0, because a nondecreasing function can have only countably many discontinuities.)

14 Prove or disprove: Let $F(t)$, $-\infty < t < \infty$, be a right-continuous nondecreasing function, $F(-\infty) = 0$, $F(+\infty) = 1$. If F has jumps at the points that form a dense set in $\mathbb{R}^1$, then the function F is a distribution function of a discrete random variable.

15 Prove or disprove: Let $F(x)$, $-\infty < x < \infty$, be a right-continuous nondecreasing function, $F(-\infty) = 0$, $F(+\infty) = 1$. If F is continuous on $\mathbb{R}^1$, it is a distribution function corresponding to a density.

16 Prove or disprove: Let $F(x)$, $-\infty < x < \infty$, be a nondecreasing function, $F(-\infty) = 0$, $F(+\infty) = 1$. If F is continuous on $\mathbb{R}^1$ and *piecewise continuously differentiable* (that is, $\mathbb{R}^1$ is divided into pieces by points $(... <) x_m < x_{m+1} < ... < x_n (< ...)$ so that F is continuously differentiable on the open intervals between these points, to the left of the smallest x_m if it exists, and to the right of the greatest of them), it is a distribution function corresponding to a density.

17 Prove or disprove: Let $F(x)$, $-\infty < x < \infty$, be a right-continuous nondecreasing function, $F(-\infty) = 0$, $F(+\infty) = 1$. If

$$\int_{-\infty}^{\infty} F'(x)\,dx = 1,$$

then the function F is a distribution function corresponding to a density.

The deadline for Problems **10**–**17** is September 26.

18 (deadline Sep. 29). Suppose we take a random permutation of numbers 1, 2, ..., n; that is, the sample space Ω consists of all $n!$ sequences $\omega = (x_1, x_2, ..., x_n)$ such that $\{x_1, x_2, ..., x_n\} = \{1, 2, ..., n\}$; (of course, $\mathcal{F} = \mathcal{P}(\Omega)$); and the probabilities are taken so that all different orders of the natural numbers from 1 to n are equally probable:

$$P\{(x_1, x_2, ..., x_n)\} = \frac{1}{n!}.$$

The random variable ξ is equal to the number of numbers i, $1 \le i \le n$, standing in their own place: for $\omega = (x_1, x_2, ..., x_n)$,

$$\xi(\omega) = \xi(x_1, x_2, ..., x_n) = \#\{i\colon x_i = i\}.$$

It was proved in the lecture that $E\xi = 1$, $E\xi^2 = 2$.

Is $E\xi^3 = 3$?

19 Let ξ_1, ξ_2 be two random variables taking the values in measurable spaces $(X_i, \mathcal{X}_i)$. Prove that $\boldsymbol{\xi} = (\xi_1, \xi_2)$ is a random vector with values in the product space $(X_1 \times X_2, \mathcal{X}_1 \times \mathcal{X}_2)$ (that is that the function $\boldsymbol{\xi}\colon \omega \mapsto \big(\xi_1(\omega), \xi_2(\omega)\big)$ is a $(\mathcal{F}, \mathcal{X}_1 \times \mathcal{X}_2)$-measurable function from Ω to $X_1 \times X_2$).

20 Prove that if real-valued random variables ξ_1, ξ_2 have (absolutely) continuous joint distribution with density $p_{\xi_1, \xi_2}(x_1, x_2)$, then each of them separately has a continuous one-dimensional distribution, with densities

$$p_{\xi_1}(x_1) = \int_{-\infty}^{\infty} p_{\xi_1, \xi_2}(x_1, x_2)\,dx_2, \qquad p_{\xi_1}(x_1) = \int_{-\infty}^{\infty} p_{\xi_1, \xi_2}(x_1, x_2)\,dx_1.$$

21 Let $f(t, \omega)$ be a $(\mathcal{B}[0, 1] \times \mathcal{F})$-measurable function on $[0, 1] \times \Omega$, taking values 0, 1. Is the following true for all such functions:

$$E \int_{[0,1]} f(t, \bullet)\,\#(dt) = \int_{[0,1]} Ef(t, \bullet)\,\#(dt),$$

i. e., $E \sum_{t \in [0,1]} f(t, \bullet) = \sum_{t \in [0,1]} Ef(t, \bullet)$?

22 Prove or disprove that for every sequence $A_1, A_2, ..., A_n, ...$ of independent events $P(\bigcap_{i=1}^{\infty} A_i) = \prod_{i=1}^{\infty} P(A_i)$.

23 Let $\xi_1, \xi_2, ..., \xi_n, ...$ and $\eta_1, \eta_2, ..., \eta_n, ...$ be two sequences of real-valued random variables; and the limits $\xi = \lim_{n\to\infty} \xi_n$, $\eta = \lim_{n\to\infty} \eta_n$ exist (exist for every $\omega \in \Omega$ – even if we omit the ω in our short notation).

Prove that if for every n the random variables ξ_n and η_n are independent, then also their limits ξ and η are independent.

(The same is true for independence of more than two random variables; the proof is the same, but the formulas in it are written longer.)

24 An easier version of Problem **23**:

Prove the same for sequences ξ_n, η_n being non-decreasing and the convergence $\xi_n(\omega) \to \xi(\omega)$, $\eta_n(\omega) \to \eta(\omega)$ $(n \to \infty)$ being uniform.

The deadline for Problems **19**–**24** is October 1.

25 Let $\xi_1, \xi_2, ..., \xi_n, ...$ be an infinite sequence of real-valued random variables. Show that the event

$$\{\overline{\lim}_{n\to\infty} \xi_n \leq x\}$$

belongs to the tail σ-algebra $\mathcal{F}_{\geq\infty}$ for every real x ($\overline{\lim}$ denotes the upper limit; another notation for it is lim sup; and we know that an upper limit, finite or infinite, always *exists*).

26 Using the fact that a finite $\lim_{n\to\infty} x_n$ exists if and only if $\lim_{n\to\infty,\, m\to\infty}(x_n - x_m) = 0$, prove that the event

$$\{\text{there exists a finite limit } \lim_{n\to\infty} \xi_n\} = \{\omega\colon \text{there exists a finite limit } \lim_{n\to\infty} \xi_n(\omega)\}$$

belongs to the tail σ-algebra.

27 Prove or disprove: the event

$$\{\lim_{n\to\infty} \xi_n = -\infty\} \in \mathcal{F}_{\geq\infty}.$$

28 Prove that the random variable $\overline{\lim}_{n\to\infty} \xi_n$ (taking values in the extended real line $[-\infty, \infty]$) is measurable with respect to the tail σ-algebra $\mathcal{F}_{\geq\infty}$.

29 Prove or disprove: the event

$$\{\lim_{n\to\infty} \frac{\xi_1 + ... + \xi_n}{n} = 0\}$$

belongs to the tail σ-algebra.

30 Prove: the event

$$\{\text{the series } \sum_{i=1}^{\infty} \xi_i \text{ converges}\}$$

belongs to $\mathcal{F}_{\geq\infty}$.

31 Prove or disprove, for nonnegative ξ_i: the random variable $\sum_{i=1}^{\infty} \xi_i$ is measurable with respect to the tail σ-algebra.

The deadline for Problems **25**–**31** is October 3.

32 (the deadline is October 10). Let a random variable ξ have the normal distribution with parameters $(0,\, b)$. Prove that all odd-numbered moments $m_k = E\xi^k$ about zero are equal to 0, and for $k = 2m$ we have:

$$m_{2m} = E\xi^{2m} = (2m-1)!! \cdot b^m,$$

where $(2m-1)!!$ denotes the product of all odd numbers from 1 to $2m-1$.

33* (* means a non-obligatory problem). Prove or disprove: if the characteristic function $f_\xi(t)$ is differentiable at 0, then $E|\xi| < \infty$.

Hint: A random variable having the standard Cauchy distribution with density $p(x) = \pi^{-1}/(1+x^2)$ has no expectation; its characteristic function $f(t) = e^{-|t|}$ is not differentiable at 0, but a little more, and it would be: at least the one-sided derivatives are not infinite. Couldn't we try and consider a density $\tilde{p}(x)$ that goes to 0 at $\pm\infty$ a little slower than $p(x)$, but still the expectation does not exist? It could be that for *this* density the characteristic function has zero derivative at $t = 0$.

Better take your density (-ies) symmetric with respect to 0 (even functions): the corresponding characteristic functions will be real-valued.

34 Let the two-dimensional random vector $\boldsymbol{xi}$ have the normal distribution with zero expectation ($E\boldsymbol{\xi} = \mathbf{0}$) and the matrix of covariances $B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the distribution of this random vector is concentrated on some line in the plane (i. e. that almost surely $c_1\xi_1 + c_2\xi_2 = \text{const}$ for some $(c_1,\, c_2) \neq (0,\, 0)$).

Deduce form this that the normal distribution with parameters $(\mathbf{0},\, B_1)$ has no density with respect to the two-dimensional Lebesgue measure λ_2.

35 Is the result of the previous problem true if we replace the matrix B_1 with $B_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$?

36 Prove that the normal distribution with parameters $(\boldsymbol{a},\, B)$, where B is a nonsingular matrix, has the density

$$p(\boldsymbol{x}) = \text{const} \cdot \exp\bigl\{-\tfrac{1}{2}\sum_{k,\,l} q_{kl}(x_k - a_k)(x_l - a_l)\bigr\},$$

where the matrix $Q = (q_{kl}) = B^{-1}$.

What is the constant in this formula equal to?

37 Let a four-dimensional random vector $(\xi_1,\, ...,\, \xi_4)$ have the normal distribution with parameters $\bigl(\mathbf{0},\, B = (b_{jk})\bigr)$. Find the fourth mixed moment $m_{1111} = E(\xi_1\xi_2\xi_3\xi_4)$.

38 Let μ_1, μ_2, ..., μ_n, ..., ν be distributions concentrated on nonnegative integers: for all natural n

$$\sum_{k=0}^{\infty} \mu_n\{k\} = \sum_{k=0}^{\infty} \nu\{k\} = 1.$$

Prove (or disprove) that if

$$\lim_{n\to\infty} \mu_n\{k\} = \nu\{k\}$$

for every $k = 0, 1, 2, 3, ...$, then $\mu_n \to_w \nu$ $(n \to \infty)$: convergence of the values of the probability mass function implies weak convergence of distributions.

39 Let μ_n be the normal distribution with parameters (a_n, b_n), $b_n > 0$; suppose $a_n \to a$, $b_n \to 0$ as $n \to \infty$.

Prove that $\mu_n \to_w \delta_a$ $(n \to \infty)$, where δ_a is the distribution concentrated at the point a: $\delta_a(C) = 1$ if $C \ni a$, and $= 0$ if $C \not\ni a$.

40 Prove or disprove: if $\mu_{\xi_n} \to_w \mu_\eta$, then $\xi_n \to \eta$ almost surely.

41 Using Theorem 13.1, prove that if $\xi_n \to_P \eta$, then $\mu_{\xi_n} \to_w \mu_\eta$.

The deadline for Problems **34**–**41** is Oct. 17.

42 Check that

$$f(t) = \begin{cases} 1 - |t|, & |t| \le 1, \\ 0, & |t| > 1, \end{cases}$$

is a characteristic function of a continuous distribution on the real line. Find its density.

43 For a random variable ξ having the density obtained in the previous problem, check that $E|\xi| = \infty$.

44 Let ξ_1, ξ_2, ..., ξ_n, ... be a sequence of independent random variables having the distribution with the density found in Problem **42** (such a sequence exists by Theorem 10.5). Let $\zeta_n = \dfrac{\xi_1 + ... + \xi_n}{n}$. Does the sequence ζ_n converge in probability to a constant as $n \to \infty$? If yes, what is this constant equal to?

45 For the sequence of random variables of the previous problem, does the weak limit of the distribution of ζ_n

$$(w) \lim_{n\to\infty} \mu_{\zeta_n}$$

exist? If yes, is the limiting distribution discrete or continuous, and what is its probability mass function or its density?

The deadline for Problems **42**–**45** is Oct. 31.

46 Give an example of a time-homogeneous Markov chain for which the limit $\lim_{k\to\infty} p_{xy}^{(k)}$ does not exist.

47 Give an example of a time-homogeneous Markov chain for which the limit $\lim_{k\to\infty} p_{xy}^{(k)}$ exists, but depends on x.

48 Is the Markov chain with the transition matrix (23 – 24.24) ergodic?

49* For finite homogeneous Markov chains, is the condition of Theorem 26.2 *necessary and sufficient* for the chain to be ergodic?

50 For the Markov chain with the transition matrix (23 – 24.24), find the solution $\boldsymbol{p} = (p_0, ..., p_4)$ of the system $\boldsymbol{p} \cdot P = \boldsymbol{p}$, $\sum_{y=0}^{4} p_y = 1$.

51 Let us change the last example in Lectures 23 – 24 taking the remainder after dividing by 6. Find the corresponding transition matrix P. Find the solution $\boldsymbol{p}$ of the system $\boldsymbol{p} \cdot P = \boldsymbol{p}$, $\boldsymbol{p} \cdot \mathbf{1} = 1$. Is $\lim_{k\to\infty} p_{xy}^{(k)} = p_y$, $x, y = 0, 1, ..., 5$?

52* For a finite ergodic Markov chain, is the solution of the system $\boldsymbol{p} \cdot P = \boldsymbol{p}$, $\boldsymbol{p} \cdot \mathbf{1} = 1$ necessarily unique?

The deadline for Problems **46** – **51** is Nov. 14.

53 (the deadline is Nov. 21). Let ξ be a continuous random variable with probability density

$$p_\xi(x) = \begin{cases} \dfrac{4 - x}{8}, & x \in [0, 4], \\ 0, & x \notin [0, 4]; \end{cases}$$

$\eta = (\xi - 1)^2$.

Find the conditional expectation $E\{\xi | \eta = y\}$.

54 (deadline Dec. 1). Invent a family of stochastic matrices P^{st}, $0 \le s \le t$, such that $P^{tt} = I$ (the identity matrix), the Chapman – Kolmogorov equation (34.9) is satisfied for all $0 \le s \le t \le u$, $P^{st} \ne \text{const}$, and P^{st} depends on (s, t) continuously.