MATH 755-01 INTRODUCTION TO PROBABILITY FALL 2008

Final test problems: Due not later than Dec. 10.

An answer only to a problem (a part of problem) is not enough: there must be some support.

1 Let $F(t)$, $-\infty < t < \infty$, be a distribution function. Let

$$\tilde{F}(t) = \sum_{x:\, x \le t} [F(x) - F(x^-)]$$

(this apparently uncountable sum is, in fact, at most countable, because there are at most countably many jump points, and the rest of the summands are 0). Prove that the function $F(t) - \tilde{F}(t)$ is non-decreasing and continuous, $\lim_{t\to\infty}\big(F(t) - \tilde{F}(t)\big) = 0$. Is it possible that $0 < \tilde{F}(+\infty) < 1$? that $\tilde{F}(+\infty) > 1$?

2 Let $(\xi,\, \eta)$ have the uniform distribution in the square $[0,\, 1]^2$. Does the random variable $\zeta = \xi^2 + \eta^2$ have a continuous distribution?

3 Let the joint distribution of the random variables ξ, η be normal with parameters $\left(\mathbf{0}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\right)$. Is the conditional distribution $P\{\xi \in C \| \eta\}$ a discrete one or a continuous one? In the first case, find the conditional probability mass function $P\{\xi = x \| \eta\}$; in the second case, find the conditional density. (And if the conditional distribution is neither discrete nor continuous, find the conditional distribution function $P\{\xi \le x \| \eta\}$.)

4 Let ξ have the normal distribution with parameters $(a,\, b)$ (a is the expectation, b, the variance). Using the equality $\xi^3 = \big(a + (\xi - a)\big)^3$, find $E\xi^3$.

5 For the random variables of Problem **3**, find $E(\xi^3 \| \eta) = E\big(\xi^3 \| \sigma(\eta)\big)$.

6 Let random variables ξ and η be independent, ξ having a discrete distribution with $P\{\xi = 0\} = P\{\xi = 2\} = P\{\xi = 4\} = 1/3$, and η having the uniform distribution on the interval $[0,\, 3]$. Find the distribution of the random variable $\xi + \eta$. Is this distribution discrete? continuous? a mixture of discrete and continuous?

7 Let ξ_1, ..., ξ_n, ... be independent random variables having the standard normal distribution; $\eta_k = (\xi_{k+1} - \xi_k)^2$. Are the random variables η_1, η_2, ..., η_n, ... independent?

Take $\zeta_n = \dfrac{\eta_1 + ... + \eta_n}{n}$. Does this sequence of random variables converge in probability to a constant? If yes, what is this constant?

8 Let ξ_1, ξ_2, ..., ξ_n, ... be independent random variables, ξ_i having Poisson distribution with parameter $1/i(i+1)$.

Does the event $\{\xi_1 + \xi_2 + ... + \xi_n + ... < \infty\}$ belong to the 'tail' σ-algebra $\mathcal{F}_{\ge +\infty}$? Is the probability of this event equal to 1? Is it equal to 0?

In the first case, what is the distribution of the random variable $\eta = \xi_1 + \xi_2 + ... + \xi_n + ...$? To be concrete: find the probability $P\{\eta = 1\}$ with accuracy up to 0.01.

In the second case: does the sequence of random variables $\xi_1 + \xi_2 + ... + \xi_n$ converge in probability to $+\infty$: i.e., is $\lim_{n\to\infty} P\{\xi_1 + \xi_2 + ... + \xi_n > C\} = 1$ for every constant C?

9 Let ξ have a uniform distribution on the interval $(0, 2)$; $\eta = \ln \xi$. Find $E\eta$. Is $E|\eta|^k < \infty$ for all k?

10 Let $\xi_1, \xi_2, ..., \xi_n, ...$ be independent random variables with uniform distribution on the interval $(0, 2)$. Which of the following events have probability 1, and which 0:

$$\big\{\lim_{n\to\infty} \prod_{i=1}^n \xi_i = 0\big\}; \qquad \big\{\lim_{n\to\infty} \prod_{i=1}^n \xi_i = +\infty\big\};$$

$$\big\{\text{there exists a finite positive limit } \lim_{n\to\infty} \prod_{i=1}^n \xi_i\big\}; \qquad \big\{\lim_{n\to\infty} \prod_{i=1}^n \xi_i = e^{3/2}\big\};$$

$$\big\{\lim_{n\to\infty} \prod_{i=1}^n \xi_i \text{ does not exist, } \varliminf_{n\to\infty} \prod_{i=1}^n \xi_i = 0,\ \varlimsup_{n\to\infty} \prod_{i=1}^n \xi_i = +\infty\big\}?$$

Hint: Take $\ln \prod_{i=1}^n \xi_i$; use the previous problem and the Strong Law of Large Numbers.

11 Check that the density $p(s, x, t, y) = \dfrac{\pi^{-1}(t-s)}{(t-s)^2 + (y-x)^2}$ satisfies the Chapman–Kolmogorov equation.

12 Let ξ_t, $0 \le t < \infty$, be a stochastic process with finite-dimensional distributions, for $0 \le t_1 \le t_2 \le ... \le t_n$,

$$P\{(\xi_{t_1}, ..., \xi_{t_n}) \in C\} = \int_X \nu(dx_0) \int_X P(0, x_0, t_1, dx_1) \int_X P(t_1, x_1, t_2, dx_2) \int_X ...$$
$$... \int_X P(t_{n-1}, x_{n-1}, t_n, dx_n)\, I_C(x_1, ..., x_n),$$

where

$$P(s, x, t, C) = \int_C \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)}\, dy, \quad 0 \le s < t,$$
$$P(t, x, t, C) = \delta_x(C).$$

For $0 \le s < t$, find the conditional distribution of the difference $\xi_t - \xi_s$ with respect to the σ-algebra $\mathcal{F}_{[0,\, s]}$ generated by the random variables ξ_v, $0 \le v \le s$.

13 Prove that the distribution of the random variable $\xi_t - \xi_s$ is normal with parameters $(0, t - s)$.

14 Prove that the random variables $\xi_s - \xi_0$, $\xi_t - \xi_s$ are independent.

15 Prove that $E|\xi_t - \xi_s|^4 = 3(t-s)^2$.