Lecture 10. More on continuous-time Markov chains.

End of proof of Theorem 9.3:

Suppose $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$.

Let us consider the moment-generating functions $\Psi_{\zeta_k}(-z) = Ee^{-z\zeta_k}$ for a fixed value of the argument $z > 0$:

$$Ee^{-z\zeta_k} = \int_0^{\infty} e^{-zx} \cdot \lambda_k e^{-\lambda_k x}\, dx = \frac{\lambda_k}{\lambda_k + z} = \frac{1}{1 + z/\lambda_k}. \tag{10.1}$$

We have, because of independence:

$$Ee^{-z(\zeta_1 + \ldots + \zeta_n)} = \prod_{k=1}^{n} Ee^{-z\zeta_k} = \prod_{k=1}^{n} \frac{1}{1 + z/\lambda_k}. \tag{10.2}$$

The limit $\lim_{n\to\infty} Ee^{-z(\zeta_1+\ldots+\zeta_n)} = \prod_{k=1}^{\infty} 1/(1 + z/\lambda_k)$. If $\lambda_k \not\to \infty$ as $k \to \infty$, there are infinitely many terms in this product that are smaller than some const < 1, and $\lim_{n\to\infty} Ee^{-z(\zeta_1+\ldots+\zeta_n)} = 0$. If $\lambda_k \to \infty$ as $k \to \infty$, then

$$\ln(1 + z/\lambda_k) = z/\lambda_k + o(1/\lambda_k) \qquad (k \to \infty), \tag{10.3}$$

$$\ln Ee^{-z(\zeta_1+\ldots+\zeta_n)} = -\sum_{k=1}^{n} [z/\lambda_k + o(1/\lambda_k)] \to -\infty \qquad (n \to \infty), \tag{10.4}$$

and $\lim_{n\to\infty} Ee^{-z(\zeta_1+\ldots+\zeta_n)} = 0$ also in this case.

Now let us use a Chebyshev-type inequality: for $C > 0$ and every natural n

$$\begin{aligned} P\Big\{\sum_{k=1}^{\infty} \zeta_k \le C\Big\} &\le P\Big\{\sum_{k=1}^{n} \zeta_k \le C\Big\} \\ &= P\Big\{\exp\Big\{-z \cdot \sum_{k=1}^{n} \zeta_k\Big\} \ge e^{-zC}\Big\} \le \frac{E\exp\{-z \cdot \sum_{k=1}^{n} \zeta_k\}}{e^{-zC}}. \end{aligned} \tag{10.5}$$

Since $\lim_{n\to\infty} Ee^{-z(\zeta_1+\ldots+\zeta_n)} = 0$, we have $P\{\sum_{k=1}^{\infty} \zeta_k \le C\} = 0$ for every positive C. Taking $C \to \infty$, we get $P\{\sum_{k=1}^{\infty} \zeta_k < \infty\} = 0$.

Theorem 10.1. *For a continuous-time Markov chain, we have*:

$$P_x\{\lim_{n\to\infty} \tau_n = \infty\} = P_x\Big\{\sum_{j=0}^{\infty} 1/v_{\eta_j} = \infty\Big\}. \tag{10.6}$$

Why could the representation (10.6) be useful? The limit $\lim_{n\to\infty} \tau_n$ is equal to the infinite sum $\sum_{k=1}^{\infty} (\tau_k - \tau_{k-1})$ of random variables that are, generally, dependent, and their dependence is through the random variables η_j ($\tau_k - \tau_{k-1}$ are *conditionally* independent with respect to the σ-algebra generated by $\eta_0, \eta_1, \eta_2, ...$). So $\sum_{k=1}^{\infty}(\tau_k - \tau_{k-1})$ has to do with both the chain $\eta_0, \eta_1, \eta_2, ...$ and the random variables $\tau_k - \tau_{k-1}$. In contrast, $\sum_{j=0}^{\infty} 1/v_{\eta_j}$ has to do with the Markov chain $\eta_0, \eta_1, \eta_2, ...$ only.

Proof. Represent the probability in the left-hand side as the expectation of the conditional probability (see formulas (2008.30.1), (2008.30.2): the generalized total probability formula):

$$P_x\{\lim_{n\to\infty} \tau_n = \infty\} = E_x P_x\{\sum_{k=1}^{\infty}(\tau_k - \tau_{k-1}) = \infty \| \eta_0, \eta_1, \eta_2, ...\}. \qquad (10.7)$$

With respect to the conditional probability $P_x\{\ \ \| \eta_0, \eta_1, \eta_2, ...\}$ the random variables $\tau_k - \tau_{k-1}$ are independent, and by the $0-1$ law the conditional probability of the series being divergent is equal to 0 or to 1. Since, conditionally, the random variables $\tau_k - \tau_{k-1}$ have exponential distributions with parameters $1/v_{\eta_{k-1}}$, we can apply Theorem 9.3, getting that the conditional probability in question is equal to 1 if and only if $\sum_{k=1}^{\infty} 1/v_{\eta_{k-1}} = \sum_{j=0}^{\infty} 1/v_{\eta_j} = \infty$. So we have:

$$P_x\{\sum_{k=1}^{\infty}(\tau_k - \tau_{k-1}) = \infty \| \eta_0, \eta_1, \eta_2, ...\} = I_{\{\sum_{j=0}^{\infty} 1/v_{\eta_j} = \infty\}}, \qquad (10.8)$$

from which we obtain (10.6).

Let us consider some examples.

For a continuous-time Markov chain on a *finite* space $X = \{x^1, ..., x^m\}$ (this is not an example, but rather a whole *class* of examples) we have for all $\omega \in \Omega$:

$$\sum_{j=0}^{n-1} 1/v_{\eta_j} \geq n \cdot 1/\max_{1\leq i\leq m} v_{x^i} \to \infty \qquad (n \to \infty), \qquad (10.9)$$

so $\lim_{n\to\infty} \tau_n = \infty$ almost surely. So for finite X Theorems 9.1, 9.2 provide a complete description of our process ξ_t, $t \geq 0$.

A concrete example: Let us consider the Markov chain on the two-point space $X = \{1, 2\}$ with transition matrix

$$P^t = \begin{pmatrix} 1/3 + 2e^{-t}/3 & 2/3 - 2e^{-t}/3 \\ 1/3 - e^{-t}/3 & 2/3 + e^{-t}/3 \end{pmatrix}, \qquad t \geq 0. \qquad (10.10)$$

It is easy to check that all entries are nonnegative; that the sum in each row is equal to 1; that P^0 is the identity matrix I; and that $P^s \cdot P^t = P^{s+t}$ (the last checking requires some calculations): all requirements that are imposed on a family of matrices to be the transition matrix of a Markov process.

Now let us find the A-matrix:

$$A = (a_{xy})_{x,\,y=1,\,2} = \big(\lim_{h\to 0+} \frac{p(t,\,x,\,y) - \delta_{xy}}{h}\big)_{x,\,y=1,\,2} = \begin{pmatrix} -2/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}. \qquad (10.11)$$

The entries at the main diagonal are $-v_x$, so $v_1 = 2/3$, $v_2 = 1/3$; the sums in each row $\sum_y a_{xy} = 0$, as it should be.

The Markov chain $\eta_0, \eta_1, \eta_2, ...$ is one with the transition matrix

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \tag{10.12}$$

this chain does not contain any randomness if we start at a non-random point: it travels $1 \to 2 \to 1 \to 2 \to ...$ if we start at the point 1 (i.e. almost surely with respect to the probability P_1), and $2 \to 1 \to 2 \to 1 \to 2 \to ...$ if we start at 2. If we start at 1, the random variables $\tau_k - \tau_{k-1}$ are independent (not just *conditionally* independent, because there is no randomness in the random variables $\eta_0, \eta_1, \eta_2, ...$) and exponential with parameters 2/3, 1/3, 2/3, 1/3, ... (and you understand what will be with respect to the probability P_2).

And here is an example of an A-matrix for which $\lim_{n\to\infty} \tau_n < \infty$ with probability 1: Let $X = \mathbb{Z}_+ = \{0, 1, 2, ...\}$, $a_{xx} = -v_x = -2^x$, $a_{x,x+1} = 2^x$, and all the rest of $a_{xy} = 0$. Here the chain $\eta_0, \eta_1, \eta_2, ...$ also does not contain any randomness: it goes on each step one step to the right. So the process ξ_t, starting, say, at 0, goes on like this: it spends an exponential time with parameter 1 at 0, then jumps to the state 1, spends there an exponential time (independent of the previous time) with parameter 2, jumps to 2, spends there an exponential time, independent from the previous two, with parameter 4, etc. And so it will go up to the time $\lim_{n\to\infty} \tau_n$, which is finite almost surely since the sum of expectations $1 + 1/2 + 1/4 + ... < \infty$ (make a picture of the graph of a trajectory that jumps from 0 to 1, from 1 to 2, from 2 to 3, etc., and makes an infinite number of jumps in the finite time interval $[0, \lim_{n\to\infty} \tau_n)$; at the point $\lim_{n\to\infty} \tau_n$ the graph will have a vertical asymptote). The process described like this is not defined for all values of t; we can define it – *if we decide what happens after the process goes to* $+\infty$.

For example, we may decide that the process jumps at that time to the state 0; then everything will start afresh: from 0 to 1, from 1 to 2, etc., spending at these points exponential times that are independent from everything that was before. At the time point $T_1 = \lim_{n\to\infty} \tau_n$ at which the jumps accumulate the right-hand limit, equal to 0, will exist, but there will be no finite left-hand limit. So it will go on and on until the second time T_2 of accumulation of jumps; at that time we take $\xi_{T_2} = 0$; the process goes on $0 \to 1 \to 2 \to ...$ until the third time T_3 of accumulation of jumps (make a picture of the trajectory travelling $0 \to 1 \to 2 \to ...$, with vertical asymptotes at the points $t = T_1$, T_2, T_3, ...). It can be proved that almost surely $\lim_{n\to\infty} T_n = \infty$, and our description almost surely yields a right-continuous trajectory $\xi_t(\omega)$ defined for all $t \in [0, \infty)$. It can be proved also that the process described like that is a Markov one.

But we could have decided that the process jumps, instead of to 0, to the point 1 after it goes to infinity; this will provide another Markov process. Also we can decide that, independently from all previous times spent at different points, after the accumulation of infinitely many jumps the process goes to a random point in $\{0, 1, 2, ...\}$ with the distribution given by $q_0, q_1, q_2, ... \geq 0$, $\sum_{x=0}^{\infty} q_x = 1$. All such processes will be Markov processes with different families of transition matrices P^t, but with the same A-matrix: $a_{xx} = -v_x = -1/2^x$, $a_{x,x+1} = 1/2^x$.

Another example: $X = \mathbb{Z}^1 = \{..., -2, -1, 0, 1, 2, 3, ...\}$; $a_{xx} = -\lambda$, $a_{x,\,x+1} = \lambda$, the rest 0, where λ is a positive constant (imagine a matrix that is infinite to the left and to the right, and up and down, with $-\lambda$ on the main diagonal and λ on the next diagonal up – and 0 at all other places). The random variables $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, ...$ are independent and exponential with parameter λ. If we start at $\xi_0 = x$, we have $\xi_t = x$ for $0 \le t < \tau_1$, $\xi_t = x + 1$ for $\tau_1 \le x < \tau_2$, in general, $\xi_t = y = x + n$ for $\tau_n \le \tau_{n+1}$. Of course, $\lim_{n\to\infty} \tau_n = \infty$ almost surely, because $\sum_{j=0}^{\infty} 1/v_{\eta_j} = \sum_{j=0}^{\infty} 1/\lambda = \infty$. We have: $p(t, x, y) = 0$ for $y < x$, and for $y = x + n \ge x$

$$p(t, x, y) = P_x\{\xi_t = y\} = P_x\{\tau_n \le t < \tau_{n+1}\} = P_x\{\tau_n \le t\} - P_x\{\tau_{n+1} \le t\}, \qquad (10.13)$$

because the events $\{\tau_{n+1} \le t\}$ and $\{\tau_n \le t < \tau_{n+1}\}$ are disjoint, and their union is the event $\{\tau_n \le t\}$.

Now let us use the fact that the random variable τ_n is a sum of n independent λ-exponential random variables (and τ_{n+1} of $n+1$ random variables).