Lecture 15. More forms of the Markov property. The local zero – one law.

We wrote several forms of the Markov property: (14.12), (14.18) (with respect to the family of σ-algebras $\mathcal{F}_{\le t}$), (14.23) (with respect to an arbitrary non-decreasing family $\mathcal{F}_t$); we can write (14.23) in the form with integrals:

$$E\big(f(\xi_u)\big\|\mathcal{F}_t\big) = P^{u-t} f(\xi_t) \tag{15.1}$$

almost surely, for every $f \in \mathbf{B}$ (or for every $f \in \mathbf{C}$ if the process moves in a metric space). To write a more general form of the Markov property, we'll need something from the set-theoretic introduction to the theory of stochastic processes.

Let us introduce a further restriction on the class of process we consider: we'll assume that for every $h \ge 0$ and every $\omega \in \Omega$ there exists a unique $\theta_h\omega \in \Omega$ such that

$$\xi_t(\theta_h\omega) = \xi_{t+h}(\omega) \tag{15.2}$$

for every $t \ge 0$.

This requirement is satisfied if the space Ω and the function $\xi_t(\omega)$ are such as we took in the proof of Kolmogorov's Theorem (Theorem 2008.35.2). Namely, it was $\Omega = X^T$ (in our case, $X^{[0,\infty)}$), the space of all functions $\omega = x_\bullet : [0, \infty) \mapsto X$ (i. e., functions on $[0, \infty)$ with values in X); and $\xi_t(\omega) = \xi_t(x_\bullet) = x_t$, the value of the function (sample point) $x_\bullet$ at the time point t. In this case, of course, $\theta_h\omega = \theta_h x_\bullet = x_{h+\bullet}$, that is the function (element of $X^{[0,\infty)}$) whose value at the point t is $(\theta_h x_\bullet)(t) = x_{h+t}$.

However we know that it's pretty inconvenient sometimes to work with the space of *all* functions, when a trajectory can be an arbitrary, very irregular, function; sometimes we restrict ourselves to dealing with processes whose all trajectories are continuous; or right-continuous; or right-continuous with limits on the left at each positive time; etc. If Ω is the space of all continuous functions $x_\bullet : [0, \infty) \mapsto X$, or of all right-continuous functions, or of all right-continuous with left limits, and $\xi_t(\omega) = \xi_t(x_\bullet) = x_t$, then the operators θ_h are also defined (because the sample space Ω goes into itself under the shift x_{h+t}). So this is not a very restrictive requirement – if we don't want to consider processes with too exotic properties of their trajectories.

The inverse-image operators θ_h^{-1} can be applied to *events* $A \subseteq \Omega$ (and in general, to subsets $A \subseteq \Omega$):

$$\theta_h^{-1} A = \{\omega\colon\ \theta_h\omega \in A\}. \tag{15.3}$$

It is easy to see that

$$\theta_h^{-1}\{\xi_t \in C\} = \{\xi_{h+t} \in C\}, \quad C \in \mathcal{X}, \tag{15.4}$$

$$\theta_h^{-1}\{(\xi_{t_1}, ..., \xi_{t_n}) \in C_n\} = \{(\xi_{h+t_1}, ..., \xi_{h+t_n}) \in C_n\}, \quad C_n \in \mathcal{X}^n. \tag{15.5}$$

It follows from this that $\theta_h^{-1}\mathcal{F}_{[0,\infty)} \subseteq \mathcal{F}_{[h,\infty)}$; that is, for every event $A \in \mathcal{F}_{[0,\infty)} = \mathcal{F}_{\ge 0}$ $= \sigma(\xi_t,\, t \ge 0)$ its inverse image $\theta_h^{-1} A$ belongs to the σ-algebra $\mathcal{F}_{[h,\infty)} = \mathcal{F}_{\ge h}$.

In other words, the mapping $\theta_h : \Omega \mapsto \Omega$ is $(\mathcal{F}_{\geq h}, \mathcal{F}_{\geq 0})$-measurable (see Lecture 2008.3).

Theorem 15.1. *If ξ_t, $t \geq 0$, is a Markov process with transition function $P(t, x, C)$ with respect to a non-decreasing family of σ-algebras $\mathcal{F}_t$, then for every event $A = \{(\xi_{t_1}, ..., \xi_{t_n}) \in C_n\}$, $C_n \in \mathcal{X}^n$, we have almost surely*:

$$P(\theta_h^{-1} A \| \mathcal{F}_h) = P_{\xi_h}(A) \tag{15.6}$$

(where $P_x(A)$ is the probability evaluated under the assumption that the process starts from the point x at time 0 – see Lecture 2, page 2 of the lecture note).

The *proof* is, essentially, that of Theorem 2008.32.5 of the previous semester. The difference is only that we did not introduce the notation P_x then; and that instead of an arbitrary non-decreasing family $\mathcal{F}_t$ of σ-algebras we considered the σ-algebras $\mathcal{F}_{\leq t}$. But repeating the old proof with the new σ-algebras does the trick.

One of the details in the proof was that the probability $P_x(A)$ (denoted then as $f_C(x)$) is $\mathcal{X}$-measurable in x for every event A of the form $A = \{(\xi_{t_1}, ..., \xi_{t_n}) \in C_n\}$, $C_n \in \mathcal{X}^n$ (see the proof in Lecture Note 2008.33).

Theorem 15.2. *For every event $A \in \mathcal{F}_{\geq 0}$ the probability $P_x(A)$ is $\mathcal{X}$-measurable in x.*

Proof. Let $\mathcal{D}$ be the class of all events A for which $P_x(A)$ is measurable; $\mathcal{D}$ is a λ-class. The class $\mathcal{C}$ of all events of the form $A = \{(\xi_{t_1}, ..., \xi_{t_n}) \in C_n\}$, $C_n \in \mathcal{X}^n$, is a π-class. According to what was said above, $\mathcal{D} \supseteq \mathcal{C}$, and by Dynkin's Lemma (Lecture Note 2008.33) $\mathcal{D} \supseteq \sigma(\mathcal{C}) = \mathcal{F}_{\geq 0}$.

Theorem 15.3. *Equality* (15.6) *holds for every event $A \in \mathcal{F}_{\geq 0}$.*

Proof. Our statement means two things: that $P_{\xi_h}(A)$ is $\mathcal{F}_h$-measurable; and that for every event $B \in \mathcal{F}_h$

$$P(B \cap \theta_h^{-1} A) = E\big(I_B \cdot P_{\xi_h}(A)\big). \tag{15.7}$$

By Theorem 15.2 the function $P_x(A)$ is $\mathcal{X}$-measurable; the random variable ξ_h is $\mathcal{F}_h$-measurable since ξ_t is adapted to $(\mathcal{F}_t)$; so the first statement holds.

Both sides in (15.7) are measures as functions of A; they coincide on the algebra of events A of the form $A = \{(\xi_{t_1}, ..., \xi_{t_n}) \in C_n\}$, so they coincide on the σ-algebra generated by this algebra, which is $\mathcal{F}_{\geq 0}$.

Now we can prove an interesting result being a form of a $0-1$ law:

Theorem 15.3. *Let ξ_t be a Markov process with respect to the family of σ-algebras $\mathcal{F}_{\leq t^+}$* (we know that if an $(\mathcal{F}_{\leq t})$-Markov process satisfies the conditions of Theorem 14.4, it is a Markov process with respect to $(\mathcal{F}_{\leq t^+})$. *Then for every event A belonging to the σ-algebra $\mathcal{F}_{\leq 0^+}$* (i. e. whose occurrence or non-occurrence can be found out on observations of the process in an arbitrarily small time interval $[0, \delta]$) *and every $x \in X$ we have either $P_x(A) = 0$, or $P_x(A) = 1$.*

For different x the probabilities $P_x(A)$ may be different.

This is called the *local* zero−one law: the $0-1$ law that we considered in Lecture 2008.11 was about σ-algebras defined as the limit as something goes to ∞, while here $\mathcal{F}_{\le 0+}$ is defined as the limit of the σ-algebras $\mathcal{F}_{\le s}$ as s goes from the right to the time point 0.

Proof. Let $A \in \mathcal{F}_{\le 0+}$. Of course, it follows from this that $A \in \mathcal{F}_{[0,\,\infty)}$. Let us apply formula (15.7) with $h = 0$, $B = A$, and the probability measure P_x as P. We have $\theta_0^{-1} A = A$, of course (θ_0 is the identity mapping). So we have:

$$P_x(A) = P_x(A \cap A) = E_x\big(I_A \cdot P_{\xi_0}(A)\big) = E_x\big(I_A \cdot P_x(A)\big) = P_x(A)^2. \tag{15.8}$$

This quadratic equation has only two solutions: $P_x(A) = 0$ and $P_x(A) = 1$.

Note that for this theorem we don't need the above requirement about the operators θ_h, because for $h = 0$ we just take instead of $\theta_h^{-1} A$ the event A itself. The θ_h-requirement was introduced in advance for some future results, not for Theorem 15.3.

Let us look at an application of the zero – one law to the Wiener process ξ_t. As the event A we take

$$A_+ = \{\text{there exists a } \delta > 0 \text{ such that } \xi_t \ge 0 \text{ for } 0 \le t \le \delta\}. \tag{15.9}$$

Clearly this event belongs to the σ-algebra $\mathcal{F}_{0+}$ (see Example 14.4, where, it's true, the time $t = 2$ was considered instead of $t = 0$, and ξ_2 instead of just 0). By Theorem 15.3, for every $x \in \mathbb{R}^1$ we have $P_x(A_+) = 0$ or $P_x(A_+) = 1$. Clearly $P_x(A_+) = 1$ for $x > 0$, and $P_x(A_+) = 0$ for $x < 0$ (make a picture of a continuous trajectory starting from the point $x > 0$, and of one starting from a point $x < 0$). But what will be for $x = 0$?

Let us consider also the event

$$A_- = \{\text{there exists a } \delta > 0 \text{ such that } \xi_t \le 0 \text{ for } 0 \le t \le \delta\}. \tag{15.10}$$

Of course, by symmetry of the Wiener process (the transition density $p(t,\,x,\,y) = \dfrac{1}{\sqrt{2\pi t}}\, e^{-(y-x)^2/2t}$ doesn't change if we change x to $-\,x$ and y to $-\,y$) we have that the probabilities $P_0(A_+)$ and $P_x(A_-)$ are the same.

We have:

$$P_0(A_+ \cup A_-) \le 1. \tag{15.11}$$

But are the events A_+ and A_- disjoint? No, we have:

$$A_+ \cap A_- = \{\text{there exists a } \delta > 0 \text{ such that } \xi_t = 0 \text{ for } 0 \le t \le \delta\} = \bigcup_{\delta > 0} \bigcap_{t \in [0,\,\delta]} \{\xi_t = 0\}. \tag{15.12}$$

Unfortunately, the unions and the intersection here are uncountable. But this is easily helped: the events $\bigcap_{t\in[0,\,\delta]}\{\xi_t = 0\}$ clearly increase as δ decreases, so we can rewrite the union of intersections as

$$\bigcup_{m=1}^{\infty} \bigcap_{t\in[0,\,1/m]} \{\xi_t = 0\}. \tag{15.13}$$

Still the intersection is an uncountable one; but this can be helped using the fact that the trajectories of the Wiener process are continuous:

$$A_+ \cap A_- = \bigcup_{m=1}^{\infty} \bigcap_{\text{rational}\, t \in [0,\, 1/m]} \{\xi_t = 0\}. \tag{15.14}$$

The intersection is not greater than any of the intersecants, so

$$P_0\Big(\bigcap_{\text{rational}\, t \in [0,\, 1/m]} \{\xi_t = 0\}\Big) \le P_0\{\xi_{1/m} = 0\} = \int_{\{0\}} p(1/m,\, 0,\, y)\; dy = 0; \tag{15.15}$$

and

$$P_0(A_+ \cap A_-) = \lim_{m \to \infty} P_0\Big(\bigcap_{\text{rational}\, t \in [0,\, 1/m]} \{\xi_t = 0\}\Big) = 0. \tag{15.16}$$

So we have, by (15.11) and (15.16):

$$P_0(A_+ \cup A_-) = P_0(A_+) + P_0(A_-) - P_0(A_+ \cap A_-) = 2P_0(A_+) \le 1, \quad P_0(A_+) \le 1/2, \tag{15.17}$$

from which it follows that

$$P_0(A_+) = P_0(A_-) = 0. \tag{15.18}$$

So almost all trajectories starting from 0 do not preserve the same sign in a small interval to the right of $t = 0$ (as it is with the functions we are accustomed to), but change their sign infinitely many times in every small right neighborhood of this point. Draw a picture of such a trajectory.