Lecture 16. Stopping times.

We are returning again to the set-theoretic introduction to the theory of stochastic processes: we are in the space $(\Omega, \mathcal{F})$, without the probability P yet.

Let $\mathcal{F}_t$, $t \in T \subseteq \mathbb{R}^1$, be a non-decreasing family of σ-algebras, $\mathcal{F}_t \subseteq \mathcal{F}$.

We say that a random variable τ taking values in $T \cup \{\infty\}$ is a *stopping time* if for every $t \in T$ the event

$$\{\tau \leq t\} \in \mathcal{F}_t. \tag{16.1}$$

Example 16.1. Every constant $\tau \equiv \text{const} = t_0 \in T \cup \{\infty\}$ is a stopping time. Indeed, the event (16.1) is either the whole Ω, or $\emptyset$.

Example 16.2. Let $T = \mathbb{Z}_+ = \{0, 1, 2, 3, ..., n, ...\}$; and let $\xi_0, \xi_1, \xi_2, ..., \xi_n, ...$ be a sequence of random variables taking values in a measurable space $(X, \mathcal{X})$. Let

$$\mathcal{F}_n = \mathcal{F}_{\leq n} = \sigma(\xi_0, \xi_1, ..., \xi_n). \tag{16.2}$$

If A is a subset of X belonging to the σ-algebra $\mathcal{X}$, then the first time of reaching this set

$$\tau = \begin{cases} \min\{n\colon\ \xi_n \in A\} & \text{if there are such } n, \\ \infty & \text{if there are no } n \text{ with } \xi_n \in A \end{cases} \tag{16.3}$$

is a stopping time with respect to the family $\mathcal{F}_{\leq n}$ of σ-algebras.

Indeed,

$$\{\tau \leq n\} = \bigcup_{i=1}^{n} \{\xi_i \in A\} \in \mathcal{F}_{\leq n}. \tag{16.4}$$

We have, in fact, considered such stopping times for Markov chains, with $A = \{y\}$.

Example 16.3. In the same situation, let

$$\sigma = \begin{cases} \sup\{n\colon\ \xi_n \in A\} & \text{if there are such } n, \\ \infty & \text{if there are no } n \text{ with } \xi_n \in A: \end{cases} \tag{16.5}$$

the *last* time of ξ_n being in A.

Again, we know something about this random time σ in the case of a Markov chain and the set A consisting of one state y: if y is recurrent, then almost surely $\sigma = \infty$, if it is transient, σ is finite with probability 1.

The random variable σ is *not* a stopping time, in general.

Indeed,

$$\{\sigma \leq n\} = \bigcup_{i=0}^{n} \{\xi_i \in A\} \cap \bigcap_{j=n+1}^{\infty} \{\xi_j \notin A\}; \tag{16.6}$$

and the events $\{\xi_j \notin A\}$, $j > n$, do *not* belong to the σ-algebra $\mathcal{F}_{\le n}$, in general. It may occur, of course, that still $\{\sigma \le n\} \in \mathcal{F}_{\le n}$ because of the special way in which the process ξ_n is constructed; but not in the general case.

Example 16.4. Let X be a metric space, $\mathcal{X} = \mathcal{B}_X$, $T = [0, \infty)$; and let all trajectories of the stochastic process ξ_t, $t \ge 0$, be continuous. Let A be a closed subset of X. Then

$$\tau = \begin{cases} \min\{t \ge 0 \colon \xi_t \in A\} & \text{if there are such } t, \\ \infty & \text{if there are no } t \text{ with } \xi_t \in A \end{cases} \tag{16.7}$$

is a stopping time with respect to the family $\mathcal{F}_{\le t}$ of σ-algebras (the minimum does exist because of the continuity; right-continuity would be enough).

Indeed,

$$\{\tau \le t\} = \bigcup_{0 \le s \le t} \{\xi_s \in A\}. \tag{16.8}$$

The union here is uncountable; but because the trajectories are continuous, we can rewrite it as

$$\{\tau \le t\} = \bigcap_{\varepsilon > 0} \bigcup_{\text{rational } s \in [0,\, t]} \{\mathrm{dist}(\xi_s,\, A) < \varepsilon\} \tag{16.9}$$

(make a picture of a trajectory with $\tau \le t$). Still not a countable number of set-theoretic operations – but it's very easy to help it:

$$\{\tau \le t\} = \bigcap_{m=1}^{\infty} \bigcup_{\text{rational } s \in [0,\, t]} \{\mathrm{dist}(\xi_s,\, A) < 1/m\}, \in \mathcal{F}_{\le t} \tag{16.10}$$

because all events $\{\mathrm{dist}(\xi_s,\, A) < 1/m\}$ here belong to $\mathcal{F}_{\le t}$.

Example 16.5. Let X be a metric space, $\mathcal{X} = \mathcal{B}_X$, $T = [0, \infty)$; and let all trajectories of the stochastic process ξ_t, $t \ge 0$, be continuous. Let A be an *open* subset of X. Then

$$\tau = \begin{cases} \inf\{t \ge 0 \colon \xi_t \in A\} & \text{if there are such } t, \\ \infty & \text{if there are no } t \text{ with } \xi_t \in A \end{cases} \tag{16.11}$$

is, generally, *not* a stopping time with respect to the family $\mathcal{F}_{\le t}$.

We can represent the event $\{\tau < t\}$ as

$$\bigcup_{\text{rational } s \in [0,\, t)} \{\xi_s \in A\}, \tag{16.12}$$

and this event belongs to the σ-algebra $\mathcal{F}_{\le t}$; but whether the event $\{\tau = t\}$ has or has not occurred cannot be determined by observation of ξ_s, $0 \le s \le t$, only: we have to be able to look in the future at least for a small time interval $[t,\, t + \delta]$. Make a picture with $A = (0,\, \infty)$, and a continuous trajectory $\xi_s(\omega)$, $0 \le s \le t$, coming from below to the level 0 at time t: we don't know whether it will turn up after that, or down (or, perhaps, as a typical trajectory of a Wiener process, will intersect the level 0 up and down infinitely many times in every interval $[t,\, t+\delta]$ of a positive length).

Of course, we are ready to accept that τ of this example is a stopping time with respect to the family of σ-algebras $\mathcal{F}_{\le t^+}$.

You have noticed that the above examples were not exactly *examples*: they were some general statements, one can say *theorems*. So I'll formulate some theorems (which can be easily converted to some classes of examples):

Theorem 16.1. *Let* $T = [0,\, \infty)$; *let* $\mathcal{F}_t,\ t \ge 0,$ *be a non-decreasing family of* σ*-algebras. Let us define*

$$\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s. \tag{16.13}$$

A random variable τ *is a stopping time with respect to the family* $\mathcal{F}_{t^+}$ *if and only if*

$$\{\tau < t\} \in \mathcal{F}_t \tag{16.14}$$

for every $t \in [0,\, \infty)$.

Proof. Let τ be a stopping time with respect to $(\mathcal{F}_{t^+})$; we want to prove that (16.14) is satisfied. For $t = 0$ there is nothing to prove: $\{\tau < 0\} = \emptyset$. For $t > 0$, $n_0 \ge 1/t$ we have:

$$\{\tau < t\} = \bigcup_{n=n_0}^{\infty} \{\tau \le t - 1/n\}. \tag{16.15}$$

Every summand here belongs to $\mathcal{F}_{(t-1/n)^+} \subseteq \mathcal{F}_t$, so the union also belongs to this σ-algebra.

Now the opposite: suppose τ satisfies (16.14); we want to prove that it is a stopping time with respect to $(\mathcal{F}_{t^+})$ – which means that

$$\{\tau \le t\} \in \mathcal{F}_{t^+}. \tag{16.16}$$

We have for every natural n_0:

$$\{\tau \le t\} = \bigcap_{n=n_0}^{\infty} \{\tau < t + 1/n\}. \tag{16.17}$$

According to (16.14), the events in the right-hand side belong to $\mathcal{F}_{t+1/n} \subseteq \mathcal{F}_{t+1/n_0}$; so $\{\tau \le t\} \in \mathcal{F}_{t+1/n_0}$. Since n_0 is an arbitrary natural number, we have

$$\{\tau \le t\} \in \bigcap_{n_0} \mathcal{F}_{t+1/n_0} = \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_{t^+}. \tag{16.18}$$

Theorem 16.2. *Let* $T = [0,\, \infty)$; *let* $\xi_t,\ t \ge 0,$ *be a stochastic process in a metric space* X *with right-continuous trajectories*; *let* τ *be defined by* (16.11), *where* A *is an open set. Then* τ *is a stopping time with respect to the* σ*-algebras* $\mathcal{F}_{t^+}$.

This is, in fact, almost the same as Example 16.5, only we assume now only the *right* continuity. Work it out yourself.

Example 16.6. The random times τ_k, $k = 0, 1, 2, ...,$ τ_y both for discrete Markov chains and for continuous-time discrete-space Markov chains that we have considered, and τ_{1y} are stopping times.

We check it "by induction", starting from smaller stopping times and going to those that are defined using these previous times.

Theorem 16.3. *If* τ *and* σ *are stopping times with respect to* $(\mathcal{F}_t)$, *then* $\min(\tau, \sigma)$ *and* $\max(\tau, \sigma)$ *are also stopping times.*

In particular, $\min(\tau, t_0)$ is a stopping time, where t_0 is a constant ($\in T$).

There is practically nothing to prove here: $\{\min(\tau, \sigma) \leq t\} = \{\tau \leq t\} \cup \{\sigma \leq t\}$, $\{\max(\tau, \sigma) \leq t\} = \{\tau \leq t\} \cap \{\sigma \leq t\}$.

Theorem 16.4. *If* τ_n *is a non-decreasing sequence of stopping times, then* $\tau_\infty = \lim_{n\to\infty} \tau_n$ *is also a stopping time.*

Proof. We have:

$$\{\tau_\infty \leq t\} = \bigcap_{n=1}^{\infty} \{\tau_n \leq t\}; \tag{16.19}$$

after which nothing remains to be proved.

In particular, for continuous-time Markov chains the first time $\tau_\infty = \lim_{n\to\infty} \tau_n$ of accumulation of an infinite number of jumps is a stopping time.

Theorem 16.5. *If* τ_n *is a non-increasing sequence of stopping times with respect to the family* $(\mathcal{F}_{t+})$, *then* $\tau_\infty = \lim_{n\to\infty} \tau_n$ *is a stopping time with respect to* $(\mathcal{F}_{t+})$.

Proof:

$$\{\tau_\infty < t\} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\tau_k < t\} \in \mathcal{F}_t; \tag{16.20}$$

and we use Theorem 16.1.

What are the stopping times introduced for? For the theory of stochastic processes – not only for that of *Markov* processes; so we will go again to where we have not only the space $(\Omega, \mathcal{F})$, but also probabilities.

Our first application, though, will be to Markov processes.

The standard Markov property means that, conditionally on the past up to time t (the *present*), the future behavior of our process, i. e. the behavior of ξ_{t+s}, $s \geq 0$, will be the same as that of the process ξ_s, $s \geq 0$, assuming that it starts afresh form the starting point ξ_t (look up the formulas expressing this in the previous lectures). Will this be so if we take as the *present* a *random* time τ?

No chance for this if τ is not a stopping time, say, if it is the *last* time that our process visits a point y (a transient state for our Markov chain). Indeed, after this moment (conditionally on what happened before this time, or unconditionally) we never return to it; while if we started afresh from the point ξ_τ, which is nothing but the point y, we have, generally, a positive probability (less than 1) to return to y.

But before we formulate what is called *the strong Markov property* we need to return once more to the set-theoretic introduction to the theory of stochastic processes. Namely, in the formulation of the Markov property we have the σ-algebra $\mathcal{F}_t$; so in the formulation of the strong Markov property with respect to the stopping time τ there should be a σ-algebra $\mathcal{F}_\tau$ associated with this stopping time; and what is this? It is *not* that we just put $\tau = \tau(\omega)$ instead of t in $\mathcal{F}_t$: $\mathcal{F}_{\tau(\omega)}$ would be a *σ-algebra depending on the sample point* ω: too exotic a mathematical object to consider. The $\mathcal{F}_\tau$ that we will be considering will be just a σ-algebra in the space Ω – but how it is defined we'll see in the next lecture.

So we are going to spend some more time in the set-theoretic introduction to the theory of stochastic processes. Also we have a couple more of examples-theorems to consider, so our next lecture will start with just $(\Omega, \mathcal{F})$, and only after that we'll go to probabilities and the stuff associated with them.