Lecture 19. Example of application of the strong Markov property in the form (18.12). Estimates for leaving a neighborhood for Markov processes.

Example 19.1. Let ξ_t, $t \geq 0$, be a one-dimensional Wiener process; let $\tau_0 = \min\{t \geq 0 \colon \xi_t = 0\}$ be the first time of its reaching 0. We want to find, for $x \in \mathbb{R}^1$, $x > 0$ the probabilities $P_x\{\tau_0 \leq u\}$, $P_x\{\tau_0 > u,\ \xi_u \in C\}$ (the last is the probability to go from x to the set $C \subseteq (0, \infty)$ in time u without touching 0). It is clear that we cannot reach the left half-line $(0, \infty)$ without touching 0 if we start from a point $\xi_0 = x > 0$.

By Theorem 18.2 we have, for a Borel $C \subseteq (0, \infty)$:

$$P_x\{\xi_u \in C \| \mathcal{F}_{\leq \tau_0}\} = P(u - \tau_0,\ \xi_{\tau_0},\ C) \tag{19.1}$$

on the event $\{\tau_0 \leq u\}$. By the generalized total probability formula we have:

$$\begin{aligned} P_x\{\tau_0 \leq u,\ \xi_u \in C\} &= E_x(I_{\{\tau_0 \leq u\}} \cdot P_x\{\xi_u \in C \| \mathcal{F}_{\leq \tau_0}\}) \\ &= E_x\big(I_{\{\tau_0 \leq u\}} \cdot P(u - \tau_0,\ \xi_{\tau_0},\ C)\big). \end{aligned} \tag{19.2}$$

In the right-hand side we have $\xi_{\tau_0} = 0$; so:

$$P_x\{\tau_0 \leq u,\ \xi_u \in C\} = E_x\big(I_{\{\tau_0 \leq u\}} \cdot P(u - \tau_0,\ 0,\ C)\big). \tag{19.3}$$

Because of the symmetry of the Wiener process we have $P(s,\ 0,\ C) = P(s,\ 0,\ -C)$, where the set $-C = \{-y \colon y \in C\}$ (the set C reflected through the mirror point 0). So

$$P_x\{\tau_0 \leq u,\ \xi_u \in C\} = E_x\big(I_{\{\tau_0 \leq u\}} \cdot P(u - \tau_0,\ 0,\ -C)\big) = P_x\{\tau_0 \leq u,\ \xi_u \in -C\}. \tag{19.4}$$

But we cannot go from $\xi_0 = x$ to the set $-C \subseteq (-\infty, 0)$ without touching 0 before the time u, so we can drop the mention of $\tau_0 \leq u$ in the last probability: $P_x\{\tau_0 \leq u,$ $\xi_u \in -C\} = P_x\{\xi_u \in -C\} = \displaystyle\int_{-C} \frac{1}{\sqrt{2\pi u}}\, e^{-(y-x)^2/2u}\, dy$: this is what the probability (19.4) is equal to. Then we have:

$$\begin{aligned} P_x\{\tau_0 > u,\ \xi_u \in C\} &= P_x\{\xi_u \in C\} - P_x\{\tau_0 \leq u,\ \xi_u \in C\} \\ &= P_x\{\xi_u \in C\} - P_x\{\xi_u \in -C\} \\ &= \int_C \frac{1}{\sqrt{2\pi u}}\, e^{-(y-x)^2/2u}\, dy - \int_{-C} \frac{1}{\sqrt{2\pi u}}\, e^{-(y-x)^2/2u}\, dy \\ &= \int_C \Big[\frac{1}{\sqrt{2\pi u}}\, e^{-(y-x)^2/2u} - \frac{1}{\sqrt{2\pi u}}\, e^{-(x+y)^2/2u}\Big]\, dy: \end{aligned} \tag{19.5}$$

the distribution of the random variable ξ_u restricted to the event $\{\tau_0 > u\}$ is described by a density (this density does not integrate to 1, but only to $P_x\{\tau_0 > u\}$).

As for the probability $P_x\{\tau_0 \le u\}$, it is equal to

$$1 - P_x\{\tau_0 > u\} = 1 - P_x\{\tau_0 > u,\, \xi_u > 0\} + P_x\{\xi_u < 0\} = 2P_x\{\xi_u < 0\}$$
$$= 2\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi u}}\, e^{-(y-x)^2/2u}\, dy = 2\int_{-\infty}^{-x/\sqrt{u}} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz. \tag{19.6}$$

In particular, from this we get: $P_x\{\tau_0 < \infty\} = \lim_{u\to\infty} P_x\{\tau_0 \le u\} = 2\int_{-\infty}^{0} = 1$, so τ_0 is almost surely finite; the probability density of this random variable

$$p_{x;\,\tau_0}(u) = \frac{d}{du}\Big[2\int_{-\infty}^{-x/\sqrt{u}} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz\Big] = \frac{x}{\sqrt{2\pi u^3}}\, e^{-x^2/2u} \tag{19.7}$$

for $u > 0$ (and, of course, $= 0$ for $u \le 0$).

By the way,

$$E_x\tau_0 = \int_0^{\infty} u \cdot \frac{x}{\sqrt{2\pi u^3}}\, e^{-x^2/2u}\, du = \infty \tag{19.8}$$

(the integral diverges at infinity, because the integrand decreases only as fast as $\mathrm{const}/u^{1/2}$).

Our next application of the strong Markov property will be to estimates of the probabilities that a Markov process leaves a neighborhood of the initial point during a specified time. We'll apply these estimates later to obtaining results about continuity (or just right continuity) of the trajectories of a Markov process based on its transition probabilities (just as in Kolmogorov's Theorem – Theorem 2008.38.1 – we deduced continuity of trajectories from the properties of *two-dimensional* distributions, namely from $E|\xi_t - \xi_s|^\alpha \le C \cdot |t - s|^{1+\beta}$).

Theorem 19.1. *Let ξ_t, $t \ge 0$, be a Markov process in a metric space X. Let ε, $h_0 > 0$, and*

$$\alpha = \sup_{x\in X,\, h\le h_0} P(h,\, x,\, \{y\colon \mathrm{dist}(y,\, x) > \varepsilon\}). \tag{19.9}$$

Then for every $x \in X$ and $t_1, \ldots, t_n \in [0,\, h_0]$

$$P_x\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\} \le 2\alpha. \tag{19.10}$$

Note that the event under the probability sign is the union $\bigcup_{i=1}^n \{\mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\}$; the best we could do if ξ_t were an arbitrary stochastic process rather than a Markov one would be the sum of the probabilities, thus getting the estimate $n \cdot \alpha$, whic grows with n; we see that for Markov processes we can do much better.

Proof. Of course,

$$P_x\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\} = P_x\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon,\, \mathrm{dist}(\xi_{h_0},\, \xi_0) \le \varepsilon\}$$
$$+ P_x\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon,\, \mathrm{dist}(\xi_{h_0},\, \xi_0) > \varepsilon\}. \tag{19.11}$$

The second summand is less or equal to

$$P_x\{\mathrm{dist}(\xi_{h_0},\, \xi_0) > \varepsilon\} = P(h_0,\, x,\, \{y\colon\, \mathrm{dist}(y,\, x) > \varepsilon\} \le \alpha. \tag{19.12}$$

To estimate the first summand, let us introduce the stopping time τ defined as

$$\tau = \begin{cases} \min\{t_i\colon\, \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\} & \text{if there are such } i,\ 1 \le i \le n, \\ h_0 & \text{otherwise.} \end{cases} \tag{19.13}$$

For the discrete stopping time τ we have

$$\begin{aligned} P_x\{\mathrm{dist}(\xi_{h_0}, \xi_0) \le \varepsilon \| \mathcal{F}_{\le\tau}\} &= P_x\bigl\{\xi_{h_0} \in \{y\colon\, \mathrm{dist}(y,\, x) \le \varepsilon\} \| \mathcal{F}_{\le\tau}\bigr\} \\ &= P\bigl(h_0 - \tau,\, \xi_\tau,\, \{y\colon\, \mathrm{dist}(y,\, x) \le \varepsilon\}\bigr) \end{aligned} \tag{19.14}$$

(on the event $\{\tau \le h_0\}$, which is, according to (19.13), the whole Ω). This means that for every event $B \in \mathcal{F}_{\le\tau}$

$$P_x\bigl(B \cap \{\mathrm{dist}(\xi_{h_0}, \xi_0) \le \varepsilon\}\bigr) = E_x\bigl(I_B \cdot P\bigl(h_0 - \tau,\, \xi_\tau,\, \{y\colon\, \mathrm{dist}(y,\, x) \le \varepsilon\}\bigr)\bigr). \tag{19.15}$$

The event $B = \{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\}$ belongs to $\mathcal{F}_{\le\tau}$ (by the time τ we know whether this event has occurred or not), so

$$\begin{aligned} &P_x\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon,\, \mathrm{dist}(\xi_{h_0}, \xi_0) \le \varepsilon\} \\ &\qquad\qquad = E_x\bigl(I_B \cdot P\bigl(h_0 - \tau,\, \xi_\tau,\, \{y\colon\, \mathrm{dist}(y,\, x) \le \varepsilon\}\bigr)\bigr). \end{aligned} \tag{19.16}$$

If the event B occurs, the point ξ_τ is outside the 2ε-neighborhood of the initial point $\xi_0 = x$; the point y in the above formula is in its ε-neighborhood; so the distance between y and ξ_τ is *greater than* ε (make a picture with the 2ε- and ε-neighborhoods of the same point $\xi_0 = x$; mark in this picture the points ξ_τ and y). So if B occurs, we have:

$$\{y\colon\, \mathrm{dist}(y,\, x) \le \varepsilon\} \subseteq \{y\colon\, \mathrm{dist}(y,\, \xi_\tau) > \varepsilon\} \tag{19.17}$$

(show this in your picture),

$$P\bigl(h_0 - \tau,\, \xi_\tau,\, \{y\colon\, \mathrm{dist}(y,\, x) \le \varepsilon\}\bigr) \le P\bigl(h_0 - \tau,\, \xi_\tau,\, \{y\colon\, \mathrm{dist}(y,\, \xi_\tau) > \varepsilon\}\bigr) \le \alpha. \tag{19.18}$$

So the probability (19.16) does not exceed $\alpha \cdot E_x I_B \le \alpha$; this estimates the first summand in (19.11), and proves our theorem.

Theorem 19.2. *Under the same conditions, let* $s \ge 0$. *Then*

$$P_x\{\max_{1\le i\le n} \mathrm{dist}(\xi_{s+t_i},\, \xi_s) > 2\varepsilon\} \le 2\alpha. \tag{19.19}$$

Proof. The event under the probability sign is $\theta_s^{-1}\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\}$, and by the ordinary Markov property with respect to the time s we have:

$$\begin{aligned} P_x\{\max_{1\le i\le n} \mathrm{dist}(\xi_{s+t_i},\, \xi_s) > 2\varepsilon\} &= E_x P_y\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\}\bigr|_{y=\xi_s} \\ &\le \sup_y P_y\{\max_{1\le i\le n} \mathrm{dist}(\xi_{t_i},\, \xi_0) > 2\varepsilon\} \le 2\alpha. \end{aligned} \tag{19.20}$$

Of course we can reformulate Theorem 19.2 like this: for $t_1, ..., t_n \in [s, s+h_0]$ (instead of the interval $[0, h_0]$)

$$P_x\{\max_{1\le i\le n} \operatorname{dist}(\xi_{t_i}, \xi_s) > 2\varepsilon\} \le 2\alpha. \tag{19.21}$$

Theorem 19.3. *Under the same conditions, let T_0 be a countable subset of the right half-line that is dense in $[0, \infty)$. Then*

$$P_x\{\sup_{t\in T_0\cap[s,\, s+h_0]} \operatorname{dist}(\xi_t, \xi_s) > 2\varepsilon\} \le 2\alpha. \tag{19.22}$$

Proof. Let $T_0 = \{t_1, t_2, ..., t_n, ...\}$; and let $T_n = \{t_1, t_2, ..., t_n\}$. Of course

$$\sup_{t\in T_0\cap[s,\, s+h_0]} \operatorname{dist}(\xi_t, \xi_s) = \lim_{n\to\infty} \max_{t\in T_n\cap[s,\, s+h_0]} \operatorname{dist}(\xi_t, \xi_s) \tag{19.23}$$

(a non-decreasing limit); so

$$\{\sup_{t\in T_0\cap[s,\, s+h_0]} \operatorname{dist}(\xi_t, \xi_s) > 2\varepsilon\} = \bigcup_{n=n_0}^{\infty} \{\max_{t\in T_n\cap[s,\, s+h_0]} \operatorname{dist}(\xi_t, \xi_s) > 2\varepsilon\}, \tag{19.24}$$

where n_0 is the natural number such that the set $T_{n_0} \cap [s, s+h_0]$ is non-empty, and

$$P_x\{\sup_{t\in T_0\cap[s,\, s+h_0]} \operatorname{dist}(\xi_t, \xi_s) > 2\varepsilon\} = \lim_{n\to\infty} P_x\{\max_{t\in T_n\cap[s,\, s+h_0]} \operatorname{dist}(\xi_t, \xi_s) > 2\varepsilon\}. \tag{19.25}$$

The set $T_n \cap [s, s+h_0]$ is finite, so the last probability is $\le 2\alpha$ by Theorem 19.2; and we get (19.22) by limit passage.

Application of this to the continuity of trajectories will be considered in the next lecture.