Lecture 20. Continuity of the trajectories. One-sided limits.

Let me first formulate some simple things that follow immediately from the results in the previous lecture.

Theorem 20.1. *Under the conditions of the theorems of the previous lecture, for every* $s \ge 0$

$$P_x\{\sup_{t,\,u\in T_0\cap[s,\,s+h_0]} \operatorname{dist}(\xi_t,\,\xi_u) > 4\varepsilon\} \le 2\alpha. \tag{20.1}$$

This is because $\operatorname{dist}(\xi_t,\,\xi_u) \le \operatorname{dist}(\xi_t,\,\xi_s)+\operatorname{dist}(\xi_u,\,\xi_s)$, and if follows from $\operatorname{dist}(\xi_t,\,\xi_u) > 4\varepsilon$ that $\operatorname{dist}(\xi_t,\,\xi_s) > 2\varepsilon$ or $\operatorname{dist}(\xi_u,\,\xi_s) > 2\varepsilon$, and $\sup_{t\in T_0\cap[s,\,s+h_0]} \operatorname{dist}(\xi_t,\,\xi_s) > 2\varepsilon$; and so

$$\{\sup_{t,\,u\in T_0\cap[s,\,s+h_0]} \operatorname{dist}(\xi_t,\,\xi_u) > 4\varepsilon\} \subseteq \{\sup_{t\in T_0\cap[s,\,s+h_0]} \operatorname{dist}(\xi_t,\,\xi_s) > 2\varepsilon\}. \tag{20.2}$$

Theorem 20.2. *Let* $t_{\max}$ *be a positive number,* $t_{\max} \le kh_0$; *then*

$$P_x\Big\{\sup_{\substack{t,\,u\in T_0\cap[0,\,t_{\max}]\\ |u-t|\le h_0}} \operatorname{dist}(\xi_t,\,\xi_u) > 8\varepsilon\Big\} \le k\cdot 2\alpha. \tag{20.3}$$

Proof. Without restriction of generality we can assume that the points ih_0 belong to the set T_0. The points t, u with $|t-u| \le h_0$ may be either in the same interval $[(i-1)h_0,\,ih_0]$, or in two different adjacent intervals, $[(i-1)h_0,\,ih_0]$ and $[ih_0,\,(i+1)h_0]$; so

$$\operatorname{dist}(\xi_t,\,\xi_u) \le \operatorname{dist}(\xi_t,\,\xi_{ih_0}) + \operatorname{dist}(\xi_{ih_0},\,\xi_u) \le 2\max_{1\le i\le n}\ \sup_{t,\,u\in[(i-1)h_0,\,ih_0]} \operatorname{dist}(\xi_t,\,\xi_u). \tag{20.4}$$

The probability

$$\begin{aligned} P_x\Big\{\sup_{\substack{t,\,u\in T_0\cap[0,\,t_{\max}]\\ |u-t|\le h_0}} \operatorname{dist}(\xi_t,\,\xi_u) > 8\varepsilon\Big\} &\le P_x\Big(\bigcup_{i=1}^{k}\{\sup_{t,\,u\in[(i-1)h_0,\,ih_0]} \operatorname{dist}(\xi_t,\,\xi_u) > 4\varepsilon\}\Big)\\ &\le \sum_{i=1}^{k} P_x\{\sup_{t,\,u\in[(i-1)h_0,\,ih_0]} \operatorname{dist}(\xi_t,\,\xi_u) > 4\varepsilon\} \le k\cdot 2\alpha. \end{aligned} \tag{20.5}$$

Now a big theorem:

Theorem 20.3. *Let* ξ_t, $t \ge 0$, *be a Markov process in a complete metric space* X. *Let its transition function satisfy the condition: for every* $\varepsilon > 0$

$$\sup_{x\in X} P(h,\,x,\,\{y\colon \operatorname{dist}(y,\,x) > \varepsilon\}) = o(h) \qquad (h \to 0^+). \tag{20.6}$$

Then there exists a stochastic process $\tilde{\xi}_t$, $t \geq 0$, *with continuous trajectories and such that for every* $x \in X$ *and* $t \geq 0$

$$P_x\{\tilde{\xi}_t = \xi_t\} = 1 \tag{20.7}$$

(and therefore $\tilde{\xi}_t$ with respect to the probabilities P_x has the same finite-dimensional distributions and is a Markov process with the same transition function).

Proof. We know that a function defined on set T_0 that is dense in $T \subseteq \mathbb{R}^1$, with values in a complete metric space, can be extended to the whole T as a continuous function if and only if it is *uniformly continuous* on the intersection of T_0 with every compact subset of T. Let T_0 be a dense countable set as in the previous theorems. Let us prove that almost surely the trajectory $\xi_t(\omega)$ is uniformly continuous on $T_0 \cap [0, t_{\max}]$ for every $t_{\max} > 0$. To do this, choose a sequence of positive $\varepsilon_n \to 0$ $(n \to \infty)$; and positive $h_n \to 0$ such that

$$\sup_{x \in X,\, h \leq h_n} P(h,\, x,\, \{y\colon \operatorname{dist}(y,\, x) > \varepsilon_n\}) < h_n/2^n. \tag{20.8}$$

Let us apply the previous theorem; the α is $< h_n/2^n$, and the integer k mentioned in it will be $\leq t_{\max}/h_n + 1$; the probability

$$P_x\Big\{\sup_{\substack{t,\, u \in T_0 \cap [0,\, t_{\max}] \\ |u-t| \leq h_n}} \operatorname{dist}(\xi_t,\, \xi_u) > 8\varepsilon_n\Big\} \leq 2(t_{\max} + h_n)/2^n. \tag{20.9}$$

The sum of these probabilities converges, so by the Borel – Cantelli Lemma almost surely for sufficiently large n $(n \geq n_0 = n_0(\omega))$ it follows from $|t - u| \leq h_n$, $t,\, u \in T_0 \cap [0,\, t_{\max}]$ that $\operatorname{dist}(\xi_t,\, \xi_u) \leq 8\varepsilon_n$. This is uniform continuity.

Now we define, for an arbitrary $t \in [0,\, \infty)$ (not necessarily $t \in T_0$)

$$\tilde{\xi}_t = \begin{cases} \lim\limits_{T_0 \ni s \to t} \xi_s & \text{if the function } \xi_s(\omega) \text{ is uniformly continuous in every } T_0 \cap [0,\, t_{\max}], \\ x_0 & \text{otherwise,} \end{cases} \tag{20.10}$$

where x_0 is an arbitrary element of X (but this alternative occurs only with zero probability). The trajectories $\tilde{\xi}_t(\omega)$ are continuous in t; what remains is proving for every t that P_x-almost surely $\tilde{\xi}_t = \xi_t$.

We have, almost surely, $\tilde{\xi}_t = \lim_{T_0 \ni s \to t} \xi_s$, from which convergence in probability follows. On the other hand, for every $\varepsilon > 0$ we have: $P_x\{\operatorname{dist}(\xi_s,\, \xi_t) > \varepsilon\} \leq \sup_{y \in X} P(|s-t|,\, y,\, \{z : \operatorname{dist}(z,\, y) > \varepsilon\}) \to 0$ as $s \to t$: ξ_t also is the limit in probability of ξ_s, $s \in T_0$, as $s \to t$. Since the limit in probability is (almost) unique, we get that $\tilde{\xi}_t = \xi_t$ almost surely.

The theorem is proved.

Unfortunately, we haven't yet developed any theory that could provide us with examples of Markov processes with continuous trajectories, apart from the Wiener process and its easy modifications. So the examples will be given later, and now a couple more theorems about the properties of trajectories.

Theorem 20.4. *Suppose a Markov process* ξ_t, $t \ge 0$, *in a complete metric space* X *satisfies the following condition*: *for every* $\varepsilon > 0$ *there exists a positive* h_0 *such that*

$$\alpha = \sup_{x \in X,\, h \le h_0} P(h,\, x,\, \{y\colon \operatorname{dist}(y,\, x) > \varepsilon\}) < 1/2. \tag{20.11}$$

Then almost surely there exist one-sided limits $\lim_{T_0 \ni s \to t^-} \xi_s$, $\lim_{T_0 \ni s \to t^+} \xi_s$ *for every* $t \in [0,\, \infty)$.

Proof. Let $\varepsilon > 0$. For every $i \ge 0$ let us introduce a sequence of random times: $\tau_0 = i h_0$,

$$\tau_k = \begin{cases} \inf\{t \in (\tau_{k-1},\, (i+1)h_0]\colon \sup\limits_{u,\, s \in T_0 \cap (\tau_{k-1},\, t]} \operatorname{dist}(\xi_s,\, \xi_u) > 4\varepsilon\} & \text{if there are such } t, \\ (i+1)h_0 & \text{otherwise.} \end{cases} \tag{20.12}$$

Let us check that τ_k are stopping times – not with respect to the σ-algebras $\mathcal{F}_{\le t}$, but rather with respect to $\mathcal{F}_{\le t^+}$.

Without loss of generality we can suppose that $(i+1)h_0$ belongs to the set T_0. Supposing that τ_{k-1} is a stopping time, let us prove that τ_k is one. To do this, we introduce

$$\sigma_k^n = \min\{t \in T_n \cap (\tau_{k-1},\, (i+1)h_0]\}, \tag{20.13}$$

$$\tau_k^n = \begin{cases} \min\{t \in T_n \cap (\tau_{k-1},\, (i+1)h_0]\colon \max\limits_{u,\, s \in T_n \cap (\tau_{k-1},\, t]} \operatorname{dist}(\xi_s,\, \xi_u) > 4\varepsilon\} \\ \qquad\qquad\qquad\qquad \text{if there are such } t, \\ (i+1)h_0 \qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{20.14}$$

where the set T_n consists of the first n elements of the countable set T_0. The random variables σ_k^n, τ_k^n are stopping times with respect to $\mathcal{F}_{\le t}$: the first one by the following theorem being a modification of Theorem 17.2:

Theorem 17.2′. *Let* $T = [0,\, \infty)$; *let* $f(t)$, $t \in [0,\, \infty]$, *be a Borel-measurable function such that* $f(t) > t$ *for every* $t \in [0,\, \infty)$, $f(\infty) = \infty$. *Let* τ *be a stopping time with respect to* $\mathcal{F}_{\le t^+}$. *Then* $\sigma = f(\tau)$ *is a stopping time with respect to* $\mathcal{F}_{\le t}$.

Of course $\lim_{n\to\infty} \tau_k^n = \tau_k$, and this is a non-increasing limit. By Theorem 16.5 we conclude that τ_k is a stopping time with respect to $\mathcal{F}_{\le t^+}$.

Now from the set-theoretic introduction, to the realm of probability. Using the strong Markov property with respect to the stopping time σ_k^n, we get:

$$P_x\{\tau_k^n < (i+1)h_0\} = P_x\big(\{\tau_{k-1} < (i+1)h_0\} \cap \{\tau_k^n < (i+1)h_0\}\big) \le P_x\{\tau_{k-1} < (i+1)h_0\} \cdot 2\alpha, \tag{20.15}$$

$$P_x\{\tau_k < (i+1)h_0\} = \lim_{n\to\infty} P_x\{\tau_k^n < (i+1)h_0\} \le P_x\{\tau_{k-1} < (i+1)h_0\} \cdot 2\alpha, \tag{20.16}$$

$$P_x\{\tau_k < (i+1)h_0\} \le (2\alpha)^k. \tag{20.17}$$

Since $\lim_{k\to\infty} P_x\{\tau_k < (i+1)h_0\} = 0$, almost surely there exist only finitely many $\tau_k < (i+1)h_0$.

So for every positive ε there are finitely many intervals $(\tau_{k-1},\, \tau_k)$ within each of which the distances between the values of ξ_t, $t \in T_0$, are not greater than 4ε.

This was in one small interval $[ih_0,\, (i+1)h_0]$; but of course the same will be for every finite interval $[0,\, t_{\max}]$.

Now we take a sequence of positive $\varepsilon_n \to 0$ $(n \to \infty)$. If the one-sided limit $\lim_{T_0 \ni s \to t^-} \xi_s$ or $\lim_{T_0 \ni s \to t^+} \xi_s$ did not exist at some point $t \in [0,\, \infty)$, there would exist an ε_n such that in every small interval $(t-\delta,\, t)$ or $(t,\, t+\delta)$ there are two points in T_0 at a distance larger than $4\varepsilon_n$; which occurs only with probability 0.

This proves the theorem.

So we can define $\tilde{\xi}_t = \lim_{s \to t^+} \xi_s$ (or $\lim_{s \to t^-} \xi_s$ for $t > 0$), and this will be a process with right-continuous trajectories having left limits (or the other way about); but the conditions of Theorem 20.4 are not enough to prove that $P_x\{\tilde{\xi}_t = \xi_t\} = 1$.