Lecture 21. Diffusion processes in $\mathbb{R}^d$.

We'll say that a Markov process ξ_t, $t \ge 0$, in a metric space X is *uniformly continuous in probability* (at $t = 0$) if for every $\varepsilon > 0$

$$P_x\{\mathrm{dist}(\xi_h, \xi_0) \ge \varepsilon\} = P\big(h, x, \{y\colon \mathrm{dist}(y, x) \ge \varepsilon\}\big) \to 0 \qquad (h \to 0^+), \tag{21.1}$$

uniformly in $x \in X$.

From uniform continuity inn probability at time $t = 0$ follows uniform continuity in probability for every other time t, uniformly also with respect to t: by the Markov property with respect to t

$$\begin{aligned} P_x\{\mathrm{dist}(\xi_{t+h}, \xi_t) \ge \varepsilon\} &= E_x P\big(h, \xi_t, \{y\colon \mathrm{dist}(y, \xi_t) \ge \varepsilon\}\big) \\ &\le \sup_{z \in X} P\big(h, z, \{y\colon \mathrm{dist}(y, z)\}\big) \to 0, \end{aligned} \tag{21.2}$$

uniformly in x and t.

Theorem 21.1. *If the Markov process ξ_t, $t \ge 0$, in a complete metric space X is uniformly continuous in probability, then there exists a Markov process $\tilde{\xi}_t$, $t \ge 0$, with right-continuous trajectories having left limits at every $t > 0$, such that $P_x\{\tilde{\xi}_t = \xi_t\} = 1$ for every t.*

Of course, there exists another process equivalent to our ξ_t, whose trajectories are *left*-continuous, with right-hand limits; but right continuity is more useful for us.

Proof. Of course if (21.1) holds uniformly in x, we have $\sup_{0 \le h \le h_0,\, x \in X} P\big(h, x, \{y\colon \mathrm{dist}(y, x)\}\big) < 1/2$ for sufficiently small positive h_0. By Theorem 20.4, almost surely the right-hand limits $\tilde{\xi}_t = \lim_{T_0 \ni s \to t^+} \xi_s$ exist for all $t > 0$, and the left-hand limits too. The new function $\tilde{\xi}_t$ is obviously right-continuous and has left limits at every point. That $\tilde{\xi}_t$ and ξ_t coincide almost surely follows from the fact that they are both limits in probability of ξ_s, $s \in T_0$, as $s \to t^+$.

I am giving some more problems about uniform continuity in probability and right continuity of the trajectories.

I told you I couldn't give any examples of Markov processes with continuous trajectories apart from the Wiener process and its easy modifications. This is not completely true if I use some results from partial differential equations.

A partial differential equation of the form

$$\begin{aligned} \frac{\partial u}{\partial t}(t, \boldsymbol{x}) &= Lu(t, \bullet)(\boldsymbol{x}) \\ &= \frac{1}{2} \sum_{i,\, j=1}^{d} a_{ij}(\boldsymbol{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}(t, \boldsymbol{x}) + \sum_{i=1}^{d} b_i(\boldsymbol{x}) \frac{\partial u}{\partial x_i}(t, \boldsymbol{x}) + c(\boldsymbol{x})\, u(t, \boldsymbol{x}), \end{aligned} \tag{21.3}$$

$t > 0$, $\boldsymbol{x} \in \mathbb{R}^d$, is called a *parabolic* equation if the matrix $\big(a_{ij}(\boldsymbol{x})\big)$ is positive definite for every $\boldsymbol{x}$. Let us introduce restrictions on the coefficients $a_{ij}(\boldsymbol{x})$, $b_i(\boldsymbol{x})$, $c(\boldsymbol{x})$:

• $a_{ij}(\boldsymbol{x})$, $b_i(\boldsymbol{x})$, $c(\boldsymbol{x})$ are bounded, and satisfy a Hölder condition:

$$|a_{ij}(\boldsymbol{y}) - a_{ij}(\boldsymbol{x})|, \quad |b_i(\boldsymbol{y}) - b_j(\boldsymbol{x})|, \quad |c(\boldsymbol{y}) - c(\boldsymbol{x})| \leq \text{const} \cdot |\boldsymbol{y} - \boldsymbol{x}|^{\alpha}, \qquad \alpha > 0; \tag{21.4}$$

• the matrix $\big(a_{ij}(\boldsymbol{x})\big)$ is uniformly positive definite:

$$\sum_{i,\,j=1}^{d} a_{ij}(\boldsymbol{x}) \cdot \lambda_i \lambda_j \geq A \cdot |\boldsymbol{\lambda}|^2 \tag{21.5}$$

for all $\boldsymbol{x}$, $\boldsymbol{\lambda}$, where A is a positive constant.

The solution of the Cauchy problem

$$\frac{\partial u}{\partial t}(t,\, \boldsymbol{x}) = Lu(t,\, \bullet)(\boldsymbol{x}), \qquad u(0,\, \boldsymbol{x}) = f(\boldsymbol{x}) \tag{21.6}$$

is not unique; but it *is* unique in the class of functions u that are bounded for $t \leq t_{\max} < \infty$, $\boldsymbol{x} \in \mathbb{R}^d$ (and even in the class of functions growing not faster than $e^{\text{const}\cdot|\boldsymbol{x}|^2}$ as $|\boldsymbol{x}| \to \infty$). For shortness, I'll be referring to functions of t and $\boldsymbol{x}$ that are bounded for $t \leq t_{\max} < \infty$, $\boldsymbol{x} \in \mathbb{R}^d$ just as *bounded.*

Theorem 21.2 (see, e. g., A.Friedman, *Partial Differential Equations of Parabolic Type*, 1964, Chap. 1, Sec. 6, pp. 23 – 24). *Under the conditions* (21.4), (21.5) *a bounded solution of the problem* (21.6) *exists for every bounded continuous initial condition* $f(\boldsymbol{x})$. *The solution* u *is nonnegative if the initial condition is nonnegative.*

(It seems to me that the point about the solution being nonnegative is not mentioned in Sec. 6 of Friedman's book: being simpler, it must be explained in one of the previous sections.)

So we can introduce linear operators P^t, $t \geq 0$, acting in the space $\mathbf{C}(\mathbb{R}^d)$ that take every bounded continuous function f to the function $P^t f = u(t,\, \bullet)$ being the "section" of the unique bounded solution u at the time level t. These operators satisfy the condition $P^0 f = f$; they satisfy also the condition $P^{t+s} f = P^t P^s f$.

Indeed, let $u(t,\, \boldsymbol{x})$ be the bounded solution of the Cauchy problem (21.6). Then for every $s \geq 0$ the function $v(t,\, \boldsymbol{x}) = u(t+s,\, \boldsymbol{x})$ is the bounded solution of the equation $\dfrac{\partial v}{\partial t} = Lv$ with the initial condition $v(0,\, \boldsymbol{x}) = u(s,\, \boldsymbol{x}) = (P^s f)(\boldsymbol{x})$. By the uniqueness, we have $v(t\,\boldsymbol{x}) = u(t+s,\, \boldsymbol{x}) = P^{t+s} f(\boldsymbol{x}) = P^t(P^s f)(\boldsymbol{x})$.

Theorem 21.3 (see the same section of the book cited: there are two theorems in it, Theorem 10 and Theorem 11). *There exists what is called a* fundamental solution *of equation* (21.3): *a continuous nonnegative function* $p(t,\, \boldsymbol{x},\, \boldsymbol{y})$, $t > 0$, $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^d$, *differentiable in* t *and twice differentiable in* $\boldsymbol{x}$, *such that*

$$\frac{\partial p}{\partial t}(t, \boldsymbol{x},\, \boldsymbol{y}) = Lp(t,\, \bullet,\, \boldsymbol{y})(\boldsymbol{x}) \tag{21.7}$$

(i. e., satisfying equation (21.3) in its first and second argument, for every $\boldsymbol{y} \in \mathbb{R}^d$). *The bounded solution of the problem* (21.6) *is written, for* $t > 0$, *as*

$$u(t,\, \boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y}) \cdot p(t,\, \boldsymbol{x},\, \boldsymbol{y})\; d\boldsymbol{y}. \tag{21.8}$$

The function $p(t,\,\boldsymbol{x},\,\boldsymbol{y})$ *satisfies the following estimate: for every* $t_{\max} < \infty$ *there exist positive constants* C *and* a *such that*

$$p(t,\,\boldsymbol{x},\,\boldsymbol{y}) \le \frac{C}{t^{d/2}}\, e^{-a|\boldsymbol{y}-\boldsymbol{x}|^2/2t} \tag{21.9}$$

for all $t \in (0,\,t_{\max}]$ *and* $\boldsymbol{x},\,\boldsymbol{y} \in \mathbb{R}^d$.

In particular, for $a_{ij}(\boldsymbol{x}) \equiv \delta_{ij}$, $b_i(\boldsymbol{x}) = c(\boldsymbol{x}) \equiv 0$, for the *heat equation* $\dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u$, the fundamental solution is the d-dimensional normal density

$$p(t,\,\boldsymbol{x},\,\boldsymbol{y}) = \frac{1}{(2\pi t)^{d/2}}\, e^{-|\boldsymbol{y}-\boldsymbol{x}|^2/2t}. \tag{21.10}$$

My notations here differ from those in Friedman's book: a different order of the arguments, coefficients depending also on time t in the book, and so the fundamental solution depending on *four* arguments; and the factor $1/2$ or no factor in formula (21.3). Probability theory and the theory of partial differential equations have different traditions.

We can introduce the function $P(t,\,\boldsymbol{x},\,C)$, $t \ge 0$, $\boldsymbol{x} \in \mathbb{R}^d$, $C \in \mathcal{B}^d$, by

$$P(t,\,\boldsymbol{x},\,C) = \begin{cases} \displaystyle\int_C p(t,\,\boldsymbol{x},\,\boldsymbol{y})\, d\boldsymbol{y}, & t > 0, \\ \delta_{\boldsymbol{x}}(C), & t = 0; \end{cases} \tag{21.11}$$

the operators $P^t f$ can be rewritten as $P^t f(\boldsymbol{x}) = \displaystyle\int_{\mathbb{R}^d} f(\boldsymbol{y})\, P(t,\,\boldsymbol{x},\,d\boldsymbol{y})$.

Clearly $P(t,\,\boldsymbol{x},\,C)$ is a measure as a function of C, and Borel measurable as a function of $\boldsymbol{x}$.

Theorem 21.4. *Let* X *be a metric space; let* P^t *be a family of linear operators defined by*

$$P^t f(x) = \int_X f(y)\, P(t,\,x,\,dy), \tag{21.12}$$

where $P(t,\,x,\,C)$, $t \ge 0$, $x \in X$, $C \in \mathcal{B}_X$, *is a function that is a measure as a function of* C, *and is* $\mathcal{B}_X$*-measurable in* x. *If* $P^{s+t} f = P^t(P^s f)$ *for* $t,\,s \ge 0$ *and every* $f \in \mathbf{C}(X)$, *then the Chapman – Kolmogorov equation* $P(s+t,\,x,\,C) = \displaystyle\int_X P(t,\,x,\,dy)\; P(s,\,y,\,C)$ *is satisfied for every Borel* C.

We'll prove this theorem in the next lecture.

So it seems that $P(t,\,\boldsymbol{x},\,C)$ satisfies the conditions that a transition function of a Markov process should satisfy. Except for the condition $P(t,\,\boldsymbol{x},\,\mathbb{R}^d) = 1$. To have this condition satisfied, we have to introduce the requirement that wasn't there in Friedman's book: $c(\boldsymbol{x}) \equiv 0$. Then for the initial condition $f(\boldsymbol{x}) = \mathbf{1}(\boldsymbol{x}) \equiv 1$ the bounded solution of (21.6) is given by $u(t,\,\boldsymbol{x}) \equiv 1$ (in (21.3), the left-hand side is equal to 0, and the partial derivatives in the right-hand side and $c(\boldsymbol{x}) \cdot u(t,\,\boldsymbol{x})$ are equal to 0). So $P(t,\,\boldsymbol{x},\,\mathbb{R}^d) = \displaystyle\int_{\mathbb{R}^d} \mathbf{1}(y)\, P(t,\,\boldsymbol{x},\,d\boldsymbol{y}) = P^t\mathbf{1}(\boldsymbol{x}) \equiv 1$.

So there exists a Markov process with transition function (21.11). Let us check that it satisfies the condition (20.6) of Theorem 20.3:

$$\begin{aligned} P(h, \boldsymbol{x}, \{\boldsymbol{y}\colon |\boldsymbol{y}-\boldsymbol{x}| > \varepsilon\}) &= \int\limits_{\{\boldsymbol{y}\colon |\boldsymbol{y}-\boldsymbol{x}|>\varepsilon\}} p(h, \boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y} \\ &\le \int\limits_{\{\boldsymbol{y}\colon |\boldsymbol{y}-\boldsymbol{x}|>\varepsilon\}} \frac{C}{h^{d/2}}\, e^{-a|\boldsymbol{y}-\boldsymbol{x}|^2/2h}\, d\boldsymbol{y} = \frac{\text{const}}{h^{d/2}} \int_\varepsilon^\infty e^{-a\rho^2/2h} \rho^{d-1}\, d\rho. \end{aligned} \tag{21.13}$$

The integral decreases exponentially fast as $h \to 0^+$, while it is divided by $h^{d/2}$: only a power of h; so the ratio decreases as $h \to 0^+$ not only as $o(h)$, but faster than *any* power of h (we don't need anything but $o(h)$, but that we do have).

So there exists a Markov process with continuous trajectories corresponding to the fundamental solution $p(t, \boldsymbol{x}, \boldsymbol{y})$ as its transition density.

Markov processes with continuous trajectories associated with linear differential operators are called *diffusion processes*; we see that with every operator L satisfying our conditions (including $c(\boldsymbol{x}) \equiv 0$) is associated a diffusion process.