Lecture 24. Existence of semigroups and Markov processes.

Example 24.1. Let P^t be the semigroup associated with the d-dimensional Wiener process:

$$P^t f(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y}) \cdot \frac{1}{(2\pi t)^{d/2}} \, e^{-|\boldsymbol{y}-\boldsymbol{x}|^2} \, d\boldsymbol{y}. \tag{24.1}$$

What is its infinitesimal operator A?

Just as in the one-dimensional case, we get that $\mathbf{C}^2_{\text{unif.}}(\mathbb{R}^d) \subseteq D_A$, and for $f \in \mathbf{C}^2_{\text{unif.}}(\mathbb{R}^d)$

$$Af(\boldsymbol{x}) = \tfrac{1}{2}\,\Delta f(\boldsymbol{x}) = \tfrac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{x}). \tag{24.2}$$

Let us return to Example 23.4 (the one-dimensional Wiener process). Can there exist a function $F \notin \mathbf{C}^1_{\text{unif.}}$ belonging to D_A? For $\lambda > 0$, let us take the function $f = AF$; according to Theorem 22.4, this function belongs to $\mathbf{B}_0 = \mathbf{C}_{\text{unif.}}$: it is bounded and uniformly continuous. Let us define the function

$$\tilde{F}(x) = \int_{-\infty}^{\infty} f(y) \cdot \frac{1}{\sqrt{2\lambda}} \, e^{-\sqrt{2\lambda}\,|y-x|} \, dy. \tag{24.3}$$

It is easy to check that this function is bounded, uniformly continuous, twice continuously differentiable, and $\tilde{F}'' = 2\lambda \tilde{F} - 2f \in \mathbf{C}_{\text{unif.}}$: $\tilde{F}$ is a function belonging to $\mathbf{C}^2_{\text{unif.}}$.

Both functions $F(x)$ and $\tilde{F}(x)$ are solutions of the equation $\lambda F - AF = f$, and we know that the solution of this equation is unique: it is $R_\lambda f$; so $F = \tilde{F} \in \mathbf{C}^2_{\text{unif.}}$.

So D_A *is* the space $\mathbf{C}^2_{\text{unif.}}$.

However, in the d-dimensional case, $d > 1$, $\mathbf{C}^2_{\text{unif.}} \neq D_A$, but $\mathbf{C}^2_{\text{unif.}} \subset D_A$: there *are* functions f for which $\Delta f(\boldsymbol{x}) = \sum_{i=1}^{d} \dfrac{\partial^2 f}{\partial x_i^2}(\boldsymbol{x})$ is continuous, while $\dfrac{\partial^2 f}{\partial x_i^2}(\boldsymbol{x})$ are not. This is a fact from *potential theory*; if we don't know it, at least we know that *we haven't proved* that $\mathbf{C}^2_{\text{unif.}} = D_A$.

So how do we find the operator A in this case?

We call a linear operator Bf, $f \in D_B \subset \mathbf{E}$, *closed* if from $f_n \in D_B$, $f_n \to f$, $Bf_n \to \varphi$ it follows that $f \in D_B$, $Bf = \varphi$. A linear operator B is closed if and only if its *graph* $\{(f, Bf) : f \in D_B\} \in \mathbf{E} \times \mathbf{E}$ is closed. Every bounded operator defined on a closed subspace is closed. Every infinitesimal operator A is closed, because $\lambda I - A$ is closed, being the inverse of a bounded, and closed, operator R_λ. The Laplace operator Δ isn't closed; but we can define the *closure* $\overline{\Delta}$ of this operator (a substantial part of the theory of partial differential equations is about how to find such closures, and how to operate with them). It turns out that the infinitesimal operator A of the multidimensional Wiener process is exactly $\frac{1}{2}\,\overline{\Delta}$.

Example 24.2. For a diffusion process associated with the linear differential operator L with coefficients satisfying the conditions of Lecture 21 we also have $D_A \supseteq \mathbf{C}^2_{\text{unif.}}$, and for $f \in \mathbf{C}^2_{\text{unif.}}$ we have $Af = Lf$.

43 Let P^t be the semigroup of linear operators on $\mathbf{B}(-\infty, \infty)$ associated with the one-dimensional Wiener process ξ_t. For a function $f \in \mathbf{B}[0, \infty)$ let us denote with $\hat{f}(x)$ the function f extended to the whole real line as an even function: $\hat{f}(x) = f(|x|)$. Let us define the operators $\hat{P}^t$ acting on the space $\mathbf{B}[0, \infty)$ by $\hat{P}^t f(x) = P^t \hat{f}(x)$, $x \in [0, \infty)$.

Check that $\hat{P}^t$, $t \geq 0$, is a one-parameter semigroup associated with a Markov transition function.

44 Let P^t be the semigroup of linear operators on $\mathbf{B}(-\infty, \infty)$ associated with the one-dimensional Wiener process ξ_t. For a function $f \in \mathbf{B}[0, \infty)$ let us denote with $\tilde{f}(x)$ the function f extended to the whole real line as a function symmetric with respect to the point $\big(0, f(0)\big)$: $\tilde{f}(x) = 2f(0) - f(-x)$ (make a picture). Let us define the operators $\tilde{P}^t$ acting on the space $\mathbf{B}[0, \infty)$ by $\tilde{P}^t f(x) = P^t \tilde{f}(x)$, $x \in [0, \infty)$.

Check that $\tilde{P}^t$, $t \geq 0$, is a one-parameter semigroup associated with a Markov transition function.

You can think also about what Markov processes correspond to these semigroups; but we don't need it now.

Example 24.3. Let us find the infinitesimal operators $\hat{A}$, $\tilde{A}$ of the semigroups $\hat{P}^t$, $\tilde{P}^t$.

Obviously, $f \in D_{\hat{A}}$ if and only if $\hat{f} \in \mathbf{C}^1_{\text{unif.}}$, and $f \in D_{\tilde{A}}$ if and only if $\tilde{f} \in \mathbf{C}^1_{\text{unif.}}$. So $D_{\hat{A}}$ consists of all functions $f \in \mathbf{C}[0, \infty)$ such that $f(x)$ is twice continuously differentiable in $(0, \infty)$ with bounded first and second derivatives having finite limits as $x \to 0^+$, and satisfying the *boundary condition* at 0: $f'(0^+) = 0$ (which we'll usually write, loosely, as $f'(0) = 0$). And for $f \in D_{\tilde{A}}$ the boundary condition is $f''(0) = 0$.

As for $\hat{A}f$, $\tilde{A}f$, they are both given by the same formula: $\hat{A}f(x) = \frac{1}{2} f''(x)$, $\tilde{A}f(x) = \frac{1}{2} f''(x)$: both operators $\hat{A}$ and $\tilde{A}$ are restrictions of the same differential operator $\frac{1}{2} \dfrac{d^2}{dx^2}$.

We see the utmost importance of the domain of definition of the infinitesimal operator: "the same" operator, only with different domains (described by boundary conditions in our case) correspond(s) to quite different Markov processes.

Now to the existence part of the Hille – Yosida theory.

I'll be denoting the Banach space in which the semigroup will be defined with $\mathbf{E}_0$; but contrary to what we had in Lecture 22, there won't be any space $\mathbf{E}$ on which the semigroup is defined and of which $\mathbf{E}_0$ is a part. As we remember, the uniqueness of the semigroup works only on the space $\mathbf{E}_0$; and the existence theorem will also be on this space. Since there is no larger space denoted with $\mathbf{E}$, we could use the notation $\mathbf{E}$ for the space on which the semigroup is constructed; but better don't let us care too much about the notations.

Theorem 24.1. *Let A be a linear operator on a Banach space $\mathbf{E}_0$ with dense domain of definition D_A. For every positive λ, let the inverse operator $R_\lambda = (\lambda I - A)^{-1}$ exist, defined on the whole $\mathbf{E}_0$, and with $\|R_\lambda\| \leq 1/\lambda$.*

Then there exists a continuous semigroup P^t, $t \geq 0$, of linear operators on $\mathbf{E}_0$, $\|P^t\| \leq 1$, with the infinitesimal operator A.

I won't give the **proof** of this theorem; let me only mention its main points. For $f \in D_A$ we have $\lambda R_\lambda f = f - R_\lambda A f$, $\|\lambda R_\lambda f - f\| \le \|Af\|/\lambda \to 0$ as $\lambda \to \infty$: $\lambda R_\lambda f \to f$. From this, using the fact that D_A is dense, we get that $\lambda R_\lambda f \to f$ for all $f \in \mathbf{E}_0$: the operators λR_λ approximate the identity operator. Now we approximate the infinitesimal operator A with the operators $A_\lambda = A \cdot \lambda R_\lambda$, which turn out to be bounded operators defined on the whole $\mathbf{E}_0$: $A \cdot \lambda R_\lambda = \lambda^2 R_\lambda - \lambda I$. Then we take the exponential function: $P_\lambda^t = e^{tA_\lambda}$; we have:

$$P_\lambda^t = e^{t\cdot\lambda^2 R_\lambda} \cdot e^{-t\cdot\lambda I} = e^{-t\lambda} \cdot e^{t\cdot\lambda^2 R_\lambda}, \tag{24.4}$$

so $\|P_\lambda^t\| = e^{-t\lambda} \cdot \|e^{t\cdot\lambda^2 R_\lambda}\| \le e^{-t\lambda} \cdot e^{t\cdot\|\lambda^2 R_\lambda\|} \le e^{-t\lambda} \cdot e^{t\lambda} = 1$ (the same as in formula (23.10) and the following).

And finally, we prove that for every $f \in \mathbf{E}_0$ there exists a limit $\lim_{\lambda\to\infty} P_\lambda^t f = P^t f$.

The condition of the inverse operator $(\lambda I - A)^{-1}$ being defined everywhere on $\mathbf{E}_0$, with norm $\le 1/\lambda$, can be reformulated as follows: for every $f \in \mathbf{E}_0$ there exists a unique solution $F \in D_A$ of the equation

$$\lambda F - AF = f; \tag{24.5}$$

and

$$\|F\| \le \|f\|/\lambda. \tag{24.6}$$

I formulated Theorem 24.1 the way it was in the original Yosida's paper. We see that this is not the way we need to apply to Markov processes: we need also

$$P^t \mathbf{1} = \mathbf{1}, \tag{24.7}$$

and

$$f \ge 0 \;\Rightarrow\; P^t f \ge 0; \tag{24.8}$$

and, which is the most important, the representation

$$P^t f(x) = \int_X f(y)\; P(t,\, x,\, dy). \tag{24.9}$$

It's not surprising that the original theorem did not have (24.7) – (24.9): the space $\mathbf{E}_0$ was not supposed to consist of *functions* at all. But (24.7), (24.8) are taken care of very easily with slight modifications of this original theorem.

Theorem 24.2. *Let the operator A and the semigroup P^t be such as in* Theorem 24.1. *Let $\mathbf{1}$ be a fixed element of the space $\mathbf{E}_0$. Then $P^t\mathbf{1} \equiv \mathbf{1}$ if and only if $A\mathbf{1} = 0$.*

Proof. If $P^t\mathbf{1} \equiv \mathbf{1}$, we have: $A\mathbf{1} = \lim_{h\to 0+} h^{-1}(P^h\mathbf{1} - \mathbf{1}) = 0$.

If $A\mathbf{1} = 0$, the solution F of the equation $\lambda F - AF = \mathbf{1}$ (see equation (24.4)) is $F = \lambda^{-1} \cdot \mathbf{1}$, so $R_\lambda \mathbf{1} = \lambda^{-1} \cdot \mathbf{1}$; $A_\lambda \mathbf{1} = A\lambda R_\lambda \mathbf{1} = A\mathbf{1} = 0$, $P_\lambda^t \mathbf{1} = e^{tA_\lambda}\mathbf{1} = \mathbf{1}$, $P^t\mathbf{1} = \lim_{\lambda\to\infty} P_\lambda^t \mathbf{1} = \mathbf{1}$.

Theorem 24.3. *Let in the space $\mathbf{E}_0$ some elements f be called nonnegative*: $f \ge 0$, *with the usual properties* (a linear combination of nonnegative elements with nonnegative coefficients is nonnegative, the limit of nonnegative elements is also nonnegative). *Then $f \ge 0 \;\Rightarrow\; P^t f \ge 0$ for all t if and only if $f \ge 0 \;\Rightarrow\; R_\lambda f \ge 0$ for all positive λ.*

Proof. the "only if" part: for $f \geq 0$ we have: $R_\lambda f = \displaystyle\int_0^\infty e^{-\lambda t} P^t f \; dt \geq 0$ as the limit of limits of nonnegative linear combinations of $P^t f \geq 0$.

The "if" part: R_λ preserves nonnegativity, so does the operator $\lambda^2 R_\lambda$ and therefore all its powers, and so do the operators $P^t_\lambda = e^{-\lambda t} \sum_{k=0}^\infty \dfrac{t^k}{k!} (\lambda^2 R_\lambda)^k$; and now we have the limit passage from P^t_λ to P^t.

The requirement $f \geq 0 \;\Rightarrow\; R_\lambda f \geq 0$ can be rewritten so: for every nonnegative $f \in \mathbf{E}_0$ the solution F of the equation (24.5) is nonnegative.

If the space $\mathbf{E}_0$ consists of functions, and the norm in it is $\|f\| = \sup_x |f(x)|$ (that is: $\mathbf{E}$ is a subspace of $\mathbf{B}(X, \mathcal{X})$: we never consider any non-measurable functions), the requirement $f \geq 0 \Rightarrow R_\lambda f \geq 0$ together with $A\mathbf{1} = 0$, $\lambda\mathbf{1} - A\mathbf{1} = \lambda\mathbf{1}$, $(\lambda I - A)^{-1}\mathbf{1} = \lambda^{-1}\mathbf{1}$ yields: for every solution of equation (24.5)

$$a \leq f(x) \leq b \text{ for all } x \;\Rightarrow\; \lambda^{-1} a \leq F \leq \lambda^{-1} b \text{ for all } x; \tag{24.10}$$

from which

$$\|F\| \leq \lambda^{-1} \|f\|. \tag{24.11}$$

Finally: we required that there should exist a *unique* solution of the equation (24.5). But now we see that it's enough to require only the *existence* of a solution for every $f \in \mathbf{E}$: its uniqueness follows automatically from (24.11).

So we can formulate the following theorem for spaces consisting of (measurable) functions, with the sup-norm:

Theorem 24.4. *Let $\mathbf{E}_0$ be some subspace of the space $\mathbf{B} = \mathbf{B}(X, \mathcal{X})$ of bounded measurable functions on a measurable space $(X, \mathcal{X})$. A linear operator A defined on $D_A \subseteq \mathbf{E}_0$ is the infinitesimal operator of a semigroup P^t, $t \geq 0$, of linear operators on $\mathbf{E}_0$ satisfying the conditions $\|P^t\| \leq 1$, $P^t\mathbf{1} = \mathbf{1}$, $f \geq 0 \;\Rightarrow\; P^t f \geq 0$, $P^h f \to f$ $(h \to 0^+)$ if and only if: the domain D_A is dense in $\mathbf{E}_0$; $\mathbf{1} \in D_A$, and $A\mathbf{1} = 0$; for every positive λ for every $f \in \mathbf{E}_0$ there exists a solution $F \in D_A$ of the equation $\lambda F - AF = f$; and from* $\inf f(x) = a$, $\sup f(x) = b$ *it follows that* $\inf F(x) \geq \lambda^{-1} a$, $\sup F(x) \leq \lambda^{-1} b$.

As I told you, the most important thing is whether the semigroup operators have an integral representation (24.9): if yes, we have a transition function $P(t, x, C)$.

Not for all spaces consisting of functions can a bounded linear operator be represented as an integral like (24.9): not for the space $\mathbf{B}(\mathbb{R}^1, \mathcal{B}^1)$, and not for $\mathbf{C}(\mathbb{R}^1)$, and not for $\mathbf{C}_{\text{unif.}}(\mathbb{R}^1)$. But there exists an important class of cases in which such representations are possible for all bounded linear operators: the spaces of continuous functions on *compact* metric spaces. Let us formulate some things that are necessary for this.

Theorem 24.5. *If X is a compact metric space, every bounded linear functional l on the space $\mathbf{C}(X)$ is represented* (uniquely) *as*

$$l(f) = \int_X f(y) \; \nu(dy), \tag{24.12}$$

where ν is a finite signed measure on $(X, \mathcal{B}_X)$.

If $f \geq 0 \;\Rightarrow\; l(f) \geq 0$, *then* ν *is just a measure* (nonnegative).

This is the (generalized) Riesz Representation Theorem; it can be found in *Real Analysis* by H.L.Royden.

It follows from this theorem that every bounded linear operator $L\colon \mathbf{C} \mapsto \mathbf{B}$ can be represented as

$$Lf(x) = \int_X f(y)\; \lambda(x,\, dy), \tag{24.13}$$

where $\lambda(x, C)$ is, for every $x \in X$, a signed measure on $\mathcal{B}_X$.

Theorem 24.6. *For every fixed* $C \in \mathcal{B}_X$ *the function* $\lambda(x, C)$ *is* $\mathcal{B}_X$*-measurable in* x.

Proof. First we consider the case of C being a closed subset of X (and so also a compact set). Take $f_n(y) = e^{-n\cdot \mathrm{dist}(y,\, C)}$; this function belongs to $\mathbf{C}(X)$, and $Lf_n \in \mathbf{B}(X)$. We have:

$$\lambda(x,\, C) = \lim_{n\to\infty} \int_X f_n(y)\; \lambda(x,\, dy) = \lim_{n\to\infty} Lf_n(x), \tag{24.14}$$

and this is measurable as the limit of a sequence of measurable functions.

Now we pass to arbitrary Borel sets C by the standard argument with π-classes and λ-classes.

If $L\mathbf{C} \subseteq \mathbf{C}$ instead of just $\subseteq \mathbf{B}$, the measure $\lambda(x, \bullet)$ is not only measurable in x, but it is weakly continuous in x.

So to every semigroup on the space $\mathbf{C}(X)$ there corresponds a Markov process in the space X. Not all Markov processes in X correspond to semigroups of operators on $\mathbf{C}(X)$, but only those that have the *Feller property* (which was, let me remind you, in the semigroup terms, that $P^t\mathbf{C}(X) \subseteq \mathbf{C}(X)$, and in the probabilistic terms, that the distribution of the random variable ξ_t evaluated under the assumption that the initial point $\xi_0 = x$ depends on x in a weakly continuous way).

But the Hille – Yosida theory allows us to handle not all semigroups, but only those that are continuous in t. We'll see in the next lecture that the continuity of the semigroup on $\mathbf{C}(X)$ is equivalent, in the case of a compact X, to uniform continuity of the corresponding Markov process in probability.

But the first thing in the next lecture will be a reformulation of the Hille – Yosida Theorem taking into account the fact that in the case of continuous functions on a compact space the supremum is just the maximum; and this is much better.