Lecture 25. Examples.

Let us reformulate Hille – Yosida Theorem for semigroups in the space of continuous functions on a compact space:

Theorem 25.1. *Let X be a compact metric space. A linear operator A defined on $D_A \subseteq \mathbf{C}(X)$ is the infinitesimal operator of a semigroup P^t on $\mathbf{C}(X)$ corresponding to a Markov process that is uniformly continuous in probability if and only if: D_A is dense in $\mathbf{C}(X)$; $\mathbf{1} \in D_A$, $A\mathbf{1} = 0$; for every sufficiently large λ for every $f \in \mathbf{C}(X)$ there exists a solution $F \in D_A$ of the equation $\lambda F - AF = f$; and if a function $F \in D_A$ reaches its maximum at some point $x \in X$, then $AF(x) \le 0$.*

The condition of AF being non-positive at a maximum point is called the *maximum principle*; it is widely used in the theory of differential equations, but it can be applied not only to differential operators (see, e. g., the integral operator considered in Example 23.1.)

Proof. In the formulation of Theorem 24.4 the infimum and the supremum were mentioned; for a compact space X they are the minimum and the maximum. The only new thing compared to that theorem is the maximum principle (the minimum is obtained from the maximum by taking everything with the minus sign).

The "only if" part: if $f(y) \le f(x)$ for all $y \in X$, we have $P^h f(x) - f(x) = \int_X [f(y) - f(x)]\ P(h, x, dy) \le 0$, and $Af(x) = \lim_{h \to 0+} h^{-1}[P^h f(x) - f(x)] \le 0$.

The "if" part: if $\lambda F - AF = f$, and the function F reaches its maximum at a point x_0, then $AF(x_0) \le 0$, and $\max_x F(x) = F(x_0) = \lambda^{-1}[f(x_0) + AF(x_0)] \le \lambda^{-1} f(x_0) \le \lambda^{-1} \max_x f(x)$.

It's easy to see that we can make the condition about the maximum a little less restrictive: *if a function $F \in D_A$ reaches its maximum at some points x, then $AF(x) \le 0$ for at least one of these points.*

Now to the relation between continuity of the semigroup P^t and uniform continuity in probability. First a theorem that holds not only for compact spaces X:

Theorem 25.2. *Let ξ_t, $t \ge 0$, be a uniformly continuous in probability Markov process on a metric space X; let $f(x)$ be a bounded uniformly continuous function on X. Then the function $P^t f$ is uniformly continuous in the sense of the norm metric in the function space:*

$$\sup\{\|P^t f - P^s f\| :\ s, t \ge 0, |t - s| \le h\} \to 0 \qquad (h \to 0). \tag{25.1}$$

The **proof** is pretty standard: we divide the integral in $(P^h f)(x) - f(x) = \int_X [f(y) - f(x)]\ P(h, x, dy)$ into one over the ε-neighborhood of x and one over its complement; etc.

In the case of the state space X being compact, *all* continuous functions are automatically uniformly continuous, so from uniform continuity of ξ_t in probability uniform continuity of $P^t f$ follows for every $f \in \mathbf{C}$.

Theorem 25.3. *Let ξ_t, $t \ge 0$, be a Markov process in a compact space X. If $\|P^h f - f\| \to 0$ as $h \to 0^+$ for every $f \in \mathbf{C}$, then the process ξ_t is uniformly continuous in probability.*

Proof. We need to prove that for every $\varepsilon > 0$ and every $\gamma > 0$ there exists a positive h_0 such that $P(h, x, \{y\colon \operatorname{dist}(y, x)\}) < \gamma$ for all $h \le h_0$ and all $x \in X$. In a compact space X for every positive ε we can find a finite ε-net (i. e., points $x_1, ..., x_N \in X$ such that for every $x \in X$ there exists a point x_i with $\operatorname{dist}(x, x_i) < \varepsilon$.

Let $x_1, ..., x_N$ be, instead of ε, a finite $(\varepsilon/3)$-net. For every x_i, $1 \le i \le N$, we define a continuous function $f_i(x)$ by

$$f_i(x) = \begin{cases} 0 & \text{if } \operatorname{dist}(x, x_i) \le \varepsilon/3, \\ \operatorname{dist}(x, x_i)/(\varepsilon/3) - 1 & \text{if } \varepsilon/3 \le \operatorname{dist}(x, x_i) \le 2\varepsilon/3, \\ 1 & \text{if } \operatorname{dist}(x, x_i) \ge 2\varepsilon/3 \end{cases} \tag{25.2}$$

(if $\operatorname{dist}(x, x_i) = \varepsilon/3$ or $2\varepsilon/3$, it's the same whichever formula we use). Since $f_i \in \mathbf{C}$, we have $\|P^h f_i - f_i\| \to 0$ $(h \to 0^+)$, and there exists an $h_i > 0$ such that $|P^h f_i(x) - f_i(x)| < \gamma$ for $h \in [0, h_i]$.

For an arbitrary $x \in X$ let us take x_i with $\operatorname{dist}(x, x_i) < \varepsilon/3$. Then $f_i(x) = 0$, and $P^h f_i(x) = \int_X f_i(y)\ P(h, x, dy) < \gamma$ for $h \le \min(h_1, ..., x_N)$. For y such that $\operatorname{dist}(y, x) \ge \varepsilon$ we have $\operatorname{dist}(y, x_i) > 2\varepsilon/3$, and so $f_i(y) = 1$; so for $h \le \min(h_1, ..., x_N)$

$$P\bigl(h, x, \{y\colon \operatorname{dist}(y, x) \ge \varepsilon\}\bigr) \le \int_X f_i(y)\ P(h, x, dy) < \gamma. \tag{25.3}$$

Example 25.1. $X = [-\infty, \infty]$ (the compactification of the real line: a metric space with the distance $\operatorname{dist}(x, y) = |\arctan x - \arctan y|$, or any other distance generating the same topology). The space $\mathbf{C}[-\infty, \infty]$ is that of all continuous functions on $[-\infty, \infty]$ – that is, functions that are continuous on $(-\infty, \infty)$ and have finite limits at ∞ and $-\infty$, which are taken to be the values $f(\infty)$, $f(-\infty)$.

Let us define a linear operator A in this space. First we tell what the domain D_A of its definition is: D_A is the set $\mathbf{C}^1[-\infty, \infty]$ of all functions belonging to $\mathbf{C}[-\infty, \infty]$ that are continuously differentiable on $(-\infty, \infty)$ with the derivative $f'(x)$ having finite limits $f'(\infty)$, $f'(-\infty)$ at $\pm\infty$.

Of course, these finite limits cannot be other than 0: if it were not 0, the function f could not have finite limits $f(\pm\infty)$.

Now the operator Af is defined by $Af(x) = f'(x)$ for $-\infty < x < \infty$, $Af(\infty) = Af(-\infty) = 0$.

Clearly $\mathbf{1} \in D_A$, $A\mathbf{1} = 0$.

Checking that D_A is dense in the space $\mathbf{C}[-\infty, \infty]$: for every function f belonging to this space, define $g(y) = f(\tan y)$, $-\pi/2 < y < \pi/2$; let us extend this function to the interval $[-\pi/2, \pi/2]$ by taking $g(-\pi/2) = f(-\infty)$, $g(\pi/2) = f(\infty)$. The function g thus defined is continuous on the closed interval $[-\pi/2, \pi/2]$. By Weierstrass' Theorem, for every $\gamma > 0$ there exists a polynomial g_γ such that $\|g_\gamma - g\| < \varepsilon$. And now we take $f_\gamma(x) = g_\gamma(\arctan x)$.

This function is differentiable on $(-\infty, \infty)$:

$$\frac{d}{dx} f_\gamma(x) = g'_\gamma(\arctan x) \cdot \frac{1}{1+x^2}, \tag{25.4}$$

and the limits of this derivative at $\pm\infty$ are equal to 0: $f_\gamma \in D_A$. And $\|f_\gamma - f\| = \|g_\gamma - g\|$ is smaller than γ. (Of course, we also get that the set $\mathbf{C}^n[-\infty, \infty]$ of functions belonging to $\mathbf{C}[-\infty, \infty]$ together with its derivatives up to order n is dense, and for infinitely differentiable too.)

If $F \in D_A$ attains its maximum at a point $x \in (-\infty, \infty)$, we know that $F'(x) = 0$, so $F'(x) \le 0$ is true; if it reaches the maximum at ∞ or $-\infty$, the same because $F'(\pm\infty) = 0$ for all $F \in \mathbf{C}[-\infty, \infty]$, whether they have a maximum at $\pm\infty$ or not..

Now to solving the equation $\lambda F - AF = f$; that is to finding a solution $F \in D_A$ of the differential equation $\lambda F - F' = f$.

Multiplying this equation by $-e^{-\lambda x}$, we get:

$$e^{-\lambda x} F'(x) - \lambda e^{-\lambda x} F(x) = \frac{d}{dx}[e^{-\lambda x} F(x)] = -e^{-\lambda x} f(x), \tag{25.5}$$

$$e^{-\lambda x} F(x) = -\int e^{-\lambda x} f(x)\, dx + C. \tag{25.6}$$

We have learned long ago that definite integrals are better than indefinite. It is possible, of course, to write this integral as a definite one, from some $a \in (-\infty, \infty)$ to x; but this depends on the choice of a and is not very convenient. It would be better if we could choose $a = -\infty$ or $+\infty$. The integral from $-\infty$ to x diverges if $f(-\infty) \ne 0$; but that from ∞ converges, and we can write:

$$e^{-\lambda x} F(x) = \int_x^\infty e^{-\lambda y} f(y)\, dy + C, \tag{25.7}$$

$$F(x) = \int_x^\infty e^{-\lambda(y-x)} f(y)\, dy + Ce^{\lambda x}. \tag{25.8}$$

The definite integral does not exceed in absolute value $\|f\| \cdot \int_x^\infty e^{-\lambda(y-x)}\, dy = \lambda^{-1} \cdot \|f\|$: bounded. But the summand $Ce^{\lambda x}$ is *not* bounded (at ∞) unless $C = 0$; so the only bounded solution (and all functions in D_A are bounded) is

$$F(x) = \int_x^\infty e^{-\lambda(y-x)} f(y)\, dy. \tag{25.9}$$

Does the function F defined by this formula belong to D_A?

It is bounded; it is continuously differentiable; do the limits at $\pm\infty$ exist?

It turns out that $F(\infty) = \lim_{x\to\infty} F(x) = \lambda^{-1} f(\infty)$, $F(-\infty) = \lim_{x\to-\infty} F(x) = \lambda^{-1} f(-\infty)$. Let us prove it.

Let $\gamma > 0$ be given; choose C so that $|f(y) - f(\infty)| < \lambda\gamma$ for $y \geq C$, $|f(y) - f(-\infty)| < \lambda\gamma/3$ for $y \leq -C$, and $e^{-\lambda C} < \lambda\gamma/3\|f\|$. Then we have:

$$|F(x) - \lambda^{-1} f(\infty)| \leq \int_x^\infty e^{-\lambda(y-x)} |f(y) - f(\infty)|\, dy < \gamma \tag{25.10}$$

for x (and therefore also y) greater than C; and

$$\begin{aligned} |F(x) - \lambda^{-1} f(-\infty)| \leq \int_x^{x+C} e^{-\lambda(y-x)} |f(y) - f(\infty)|\, dy + \int_{x+C}^\infty e^{-\lambda(y-x)} \cdot 2\|f\|\, dy \\ < \gamma/3 + 2\gamma/3 = \gamma \end{aligned} \tag{25.11}$$

for $x \leq -2C$.

This means that $F \in \mathbf{C}[-\infty, \infty]$. To prove that $F \in \mathbf{C}^1[-\infty, \infty]$, we have to check that $F' \in \mathbf{C}[-\infty, \infty]$. But this follows from $F' = \lambda F - f$.

So the Markov process corresponding to our operator A does exist: we recognize it as the deterministic motion $\xi_t = \xi_0 + t$ (only in Lecture 23 we found the infinitesimal operator of the semigroup on the whole space $\mathbf{B}(-\infty, \infty)$, and here we consider the restriction of this semigroup to the space of continuous functions with limits at $\pm\infty$). The only thing that wasn't there before is: what do the trajectories do if the initial point is $\xi_0 = \infty$ or $-\infty$? It can be proved that a trajectory starting at the point ∞ remains in this point forever; and the same for $-\infty$.

45 Let the semigroup P^t corresponding to a Markov process ξ_t in a metric space X take the space $\mathbf{C}(X)$ to $\mathbf{C}(X)$, and let P^t be continuous on $\mathbf{C}(X)$. Let x_0 be a point such that $Af(x_0) = 0$ for every $f \in D_A$.

Using the fact that $\dfrac{d}{dt} P^t f = P^t A f = A P^t f$ for $f \in D_A$, prove that $P^t f(x_0) = f(x_0)$ for every $f \in D_A$.

What equality did you use: $\dfrac{d}{dt} P^t f = P^t A f$ or $\dfrac{d}{dt} P^t f = A P^t f$?

46 Under the same conditions, prove $P^t f(x_0) = f(x_0)$ for every $f \in \mathbf{C}(X)$.

It follows from this that $P(t, x_0, \{x_0\}) = 1$: the process almost surely does not leave the point x_0 if it starts from it (or, by the strong Markov property, if it ever gets to the point x_0).

Example 25.2. $X = [-\infty, \infty]$, $D_A = \mathbf{C}^2[-\infty, \infty]$, $Af = f''$.

That D_A is dense: same reasoning; the maximum principle: if a function $F \in \mathbf{C}^2[-\infty, \infty]$ reaches its maximum at $x \in (-\infty, \infty)$, then the first derivative $F'(x)$ is equal to 0, and the second $F''(x) \leq 0$ (this much we know from our Calculus I); and at the points $\pm\infty$ the same way as in Example 25.1. Solving the equation $\lambda F - F'' = f$: the solution too has an integral representation (see formula (24.3) that was written not for the operator of second differentiation, but for its half). The corresponding Markov process is, up to some minor changes, the Wiener process (with staying forever at the points $\pm\infty$).

Example 25.3. $X = [-\infty, \infty]$, $D_A = \mathbf{C}^3[-\infty, \infty]$, $Af = f'''$.

The denseness of D_A is the same, but the maximum principle does *not* hold: say, the function $F(x) = \dfrac{1}{1+x^2} + C \cdot \dfrac{x^3}{1+x^6}$ reaches its maximum at $x = 0$ for sufficiently small C, but its third derivative at this point is equal to $6C$, which is positive if $C > 0$.

So: no Markov processes corresponding to the third derivative (or to higher derivatives).

We have to restrict ourselves to the first and second derivatives. Let us consider examples with the *first* derivative: on the one hand, it's simpler, and on the other it, strangely enough, offers more diversity.

Example 25.4. $X = [0, \infty]$, D_A is the set $\mathbf{C}[0, \infty]$ of all continuous functions on $[0, \infty]$ that are continuously differentiable on $(0, \infty)$ with the derivative $f'(x)$ having finite limits $f'(0)$ (which is the short notation for the limit $f'(0^+)$), $f'(\infty)$ $(= 0)$; $Af(x) = f'(x)$.

The same reasoning about D_A being dense; if $F \in D_A$ reaches its maximum at $x \in (0, \infty]$, we have $F'(x) = 0$; and if it reaches the maximum at $x = 0$, then $F'(0) \leq 0$ (make a picture of a function on the right half-line reaching its maximum at 0).

Finally, the formula (25.9) still provides a solution $F \in D_A$ of the equation $\lambda F - AF = f$.

The corresponding process is again the deterministic motion to the right at speed 1, including the case of the initial point $\xi_0 = 0$.

Example 25.5. $X = [-\infty, 0]$, D_A is the set $\mathbf{C}^1[-\infty, 0]$ of all continuous functions on $[-\infty, 0]$ that are continuously differentiable on $(-\infty, 0)$ with the derivative $f'(x)$ having finite limits $f'(-\infty) = 0$ and $f'(0^-)$, which we'll denote just as $f'(0)$; and $Af(x) = f'(x)$.

We feel apprehensive from the very beginning: the motion $\xi_t = \xi_0 + t$, $t \geq 0$, won't work on the left half-line: we would reach 0, and what then? But still let us check whether the conditions of our theorem are satisfied.

No difference at all with D_A being dense; as for the maximum condition, we see that from $F(x_0) = \max_{x \in (-\infty, 0]} F(x)$ it *doesn't* follow that $F'(x_0) \leq 0$: make a picture of a smooth function on $[-\infty, 0]$ reaching its maximum at $x_0 = 0$ and having a positive left-hand limit $F'(0)$ (the function e^x will do).

So there is *no* Markov process with A as its infinitesimal operator.

Example 25.6. $X = [-\infty, 0]$, D_A is the set of functions $f \in \mathbf{C}^1[-\infty, 0]$ with $f'(0) = 0$; $Af(x) = f'(x)$.

The condition $f'(0) = 0$ is a *boundary condition* (we remember that sometimes when formulating problems for partial differential equations we introduce boundary conditions).

Again, D_A is dense in $\mathbf{C}[-\infty, 0]$. Here the proof is more complicated: we define $g(y) = f(\tan y)$, $y \in (-\pi/2, 0]$, and we take $g(-\pi/2) = f(-\infty)$. This is a continuous function on $[-\pi/2, 0]$. But if we take Weierstrass' polynomial $g_\varepsilon(y)$, its derivative at 0 won't necessarily be equal to 0; we want to repair it.

Let us consider the linear functional l defined on $D_l = \mathbf{C}^1[-\infty, 0]$ by $l(f) = f'(0)$. This functional is unbounded with respect to the norm $\|\ \|$: there exist functions $f_n(x) = \arctan(nx)$ with $\|f_n\| = \pi/2$, whereas $l(f_n) = n$ can be arbitrarily large.

Theorem 25.4. *Let* $\mathbf{E}$ *be a normed space,* l *a nonzero bounded linear functional with the domain of definition* D_l*. Then the set* $\{f \in D_l \colon l(f) = 0\}$ *is* not *dense in* $\mathbf{E}$.

The **proof** is very easy: since l is not the identical zero, there exists an $f_0 \in \mathbf{E}$ such that $l(f_0) \neq 0$. Then there exists a neighborhood of the point f_0 that has no points f with $l(f) = 0$ in it (the boundedness of the functional l is used).

Theorem 25.5. *Let* $\mathbf{E}$ *be a normed space,* l *a nonzero linear functional with its domain* D_l *dense in* $\mathbf{E}$.

If the functional l *is unbounded, the set* $\{f \in D_l \colon l(f) = 0\}$ *is dense in* $\mathbf{E}$.

Proof. We want to prove that for every $f \in \mathbf{E}$ and every $\gamma > 0$ there exists an element $\hat{f}_\gamma$ with $l(\hat{f}_\gamma) = 0$ such that $\|\hat{f}_\gamma - f\| < \gamma$. We do it like this: since D_l is dense, there exists an $f_\gamma \in D_l$, $\|f_\gamma - f\| < \gamma/2$; but $l(f_\gamma)$ may be not equal to 0.

Since the functional l is unbounded, for every $C > 0$ there is a $g_C \in D_l$ with $\|g_C\| = 1$, $|l(g_C)| > C$. Take $C = 1\|l(f_\gamma)|/\gamma$, take the corresponding g_C, and $\hat{f}_\gamma = f_\gamma - \dfrac{l(f_\gamma)}{l(g_C)} \cdot g_C$. We have: $\|\hat{f}_\gamma - f\| \le \|\hat{f}_\gamma - f_\gamma\| + \|f_\gamma - f\| = |\dfrac{l(f_\gamma)}{l(g_C)}| \cdot \|g_C\| + \dfrac{\gamma}{2} \le \dfrac{\gamma}{2} + \dfrac{\gamma}{2} = \gamma$; $l(\hat{f}_\gamma) = l(f_\gamma) - \dfrac{l(f_\gamma)}{l(g_C)} \cdot l(g_C) = 0$.

Applying Theorem 25.5 to Example 25.6, we get that D_A is dense in $\mathbf{C}[-\infty, 0]$.

Now to the maximum principle: if $F \in D_A$ reaches its maximum at a point $x \in [-\infty, 0]$, we have $F'(x) = 0$ because of the general property at a maximum (for $x > -\infty$), or because of boundedness of F for $x = -\infty$; and for $x = 0$ because of the boundary condition $F'(0) = 0$ that is required from $F \in D_A$.

What about solving the equation $\lambda F - AF = f$, $f \in \mathbf{C}[-\infty, 0]$? Let us take

$$\tilde{F}(x) = \int_x^0 e^{-\lambda(y-x)} f(y)\, dy. \tag{25.12}$$

Straightforward differentiation shows that $\lambda\tilde{F} - \tilde{F}' = f$; and $\lim_{x\to-\infty} \tilde{F}(x) = \lambda^{-1} f(-\infty)$. However in general, the boundary condition $\tilde{F}'(0) = 0$ is *not* satisfied. But the general solution of the equation $\lambda F - F' = f$ is equal to its particular solution plus the general solution of the corresponding homogeneous equation:

$$F(x) = \tilde{F}(x) + C \cdot e^{\lambda x}; \tag{25.13}$$

the function $e^{\lambda x}$ is bounded and continuous on $[-\infty, 0]$ together with its derivative. What remains is to choose the constant C so that $F'(0) = 0$.

We have:

$$F'(0) = \tilde{F}'(0) + C \cdot \frac{d}{dx}(e^{\lambda x})\big|_{x=0} = \tilde{F}'(0) + C \cdot \lambda. \tag{25.14}$$

Choosing $C = -\lambda^{-1} \cdot \tilde{F}'(0)$, we obtain the function $F \in D_A$.

Here we are again with a Markov process whose infinitesimal operator is A.

It's interesting to try to understand what happens with the process ξ_t after the time that it, moving (presumably) to the right at speed 1 reaches the end 0 of our interval.