Lecture 29. The Wiener process.

Example 29.1. Let ξ_t be a (one-dimensional) Wiener process starting from a point $\xi_0 = x$ at time 0; let (a, b) be an interval containing the point x. If we want to find the expectation $E_x \tau_{(a,\,b)}$ (and to find out whether this expectation is finite), the way to do it is to find this expectation (denote it $m(x)$) at once for *all* points $x \in (a,\, b)$ (no problem finding this expectation for $x = a$ or b: it is equal to 0, because almost surely with respect to P_a and to P_b we have $\tau_{(a,\,b)} = 0$). We look for $m(x)$ as the solution of the boundary-value problem

$$\begin{aligned} &\frac{1}{2}\, m''(x) = -1, \qquad a < x < b, \\ &m(a) = m(b) = 0. \end{aligned} \tag{29.1}$$

The solution is easily found:

$$m(x) = (b - x)(x - a), \qquad a \le x \le b \tag{29.2}$$

(check that this is a solution).

Formula (29.2) defines a function on the whole real line, and this function grows not faster than exponentially. By Theorem 28.5 the expectation $E_x \tau_{(a,\,b)}$ is given by (29.2).

So $\tau = \tau_{(a,\,b)}$ is finite almost surely, and we can speak of the distribution of the exit point ξ_τ: the probabilities $P_x\{\xi_\tau = a\}$, $P_x\{\xi_\tau = b\}$.

By Theorem 28.6, we should look for the expectation $u(x) = E_x \varphi(\xi_\tau)$ as the solution of the boundary-value problem

$$\begin{aligned} &\frac{1}{2}\, u''(x) = 0, \qquad x \in (a, b), \\ &u(a) = \varphi(a), \quad u(b) = \varphi(b). \end{aligned} \tag{29.3}$$

The second derivative is equal to 0: the function is linear. It is found easily:

$$u(x) = \varphi(a) \cdot \frac{b - x}{b - a} + \varphi(b) \cdot \frac{x - a}{b - a}, \qquad a \le x \le b \tag{29.4}$$

(check it). Again a function is defined by this formula in the whole real line, growing slower than exponentially. So the expectation $u(x) = E_x \varphi(\xi_\tau)$ is given by (29.4). On the other hand,

$$E_x \varphi(\xi_\tau) = \varphi(a) \cdot P_x\{\xi_\tau = a\} + \varphi(b) \cdot P_x\{\xi_\tau = b\}. \tag{29.5}$$

Using the fact that $P_x\{\xi_\tau = a\} + P_x\{\xi_\tau = b\} = 1$, we get one linear algebraic equation with one unknown, say, $P_x\{\xi_\tau = b\}$, which has a unique solution. So we get:

$$P_x\{\xi_\tau = a\} = \frac{b - x}{b - a}, \qquad P_x\{\xi_\tau = b\} = \frac{x - a}{b - a}, \qquad a \le x \le b. \tag{29.6}$$

This is the general situation: if we can solve the problem (28.25) for a sufficiently wide class of boundary functions φ, we find the distribution of the boundary point $\boldsymbol{\xi}_\tau$ at which the process $\boldsymbol{\xi}_t$ leaves the region G.

We can also consider the times $\tau_{(a,\,\infty)}$, $\tau_{(-\,\infty,\,b)}$ of leaving an infinite interval containing the initial point x. Are these times almost surely finite? (If they are, there is no question about $\xi_{\tau_{(a,\,\infty)}}$, $\xi_{\tau_{(-\,\infty,\,b)}}$: the process can leave a semi-infinite interval in no other way than through its only end.)

We have for all $\omega \in \Omega$:

$$\tau_{(a,\,\infty)} = \lim_{b\to\infty} \tau_{(a,\,b)}. \tag{29.7}$$

As for the event $\{\tau_{(a,\,\infty)} < \infty\}$, it is represented as

$$\{\tau_{(a,\,\infty)} < \infty\} = \bigcup_{b>x} \{\xi_{\tau_{(a,\,b)}} = a\}. \tag{29.8}$$

Indeed, if for some $\omega \in \Omega$ the event in the left-hand side occurs, ξ_t leaves the interval $(a,\,b)$ at the time $\tau_{(a,\,\infty)}$ through its left end (make a picture). The maximum $\max_{0\le t\le\tau_{(a,\,\infty)}} \xi_t$ is finite, and for b greater than this maximum the event $\{\xi_{\tau_{(a,\,b)}} = a\}$ occurs. So $\{\tau_{(a,\,\infty)} < \infty\} \subseteq \bigcup_{b>x} \{\xi_{\tau_{(a,\,b)}} = a\}$.

The opposite inclusion: if the event in the right-hand side of (29.8) occurs, then for some $b \ge x$ the event $\{\xi_{\tau^x_{(a,\,b)}} = a\}$ occurs; and at some time the Wiener process *is* at the point a: we have left the interval $(a,\,\infty)$.

The family of events in the right-hand side of (29.7) is non-decreasing ($\{\xi_{\tau_{(a,\,b_1)}} = a\} \subseteq \{\xi_{\tau_{(a,\,b_2)}} = a\}$ for $b_2 > b_1$), so

$$P_x\{\tau_{(a,\,\infty)} < \infty\} = \lim_{b\to\infty} P_x\{\xi_{\tau_{(a,\,b)}} = a\} = \lim_{b\to\infty} \frac{b-x}{b-a} = 1. \tag{29.9}$$

So the Wiener process *does* leave almost surely every semi-finite interval, it does almost surely reach every point a to the left of the starting point x, and every point b to the right of it. It follows from this that almost surely

$$\underline{\lim}_{t\to\infty} \xi_t = -\,\infty, \qquad \overline{\lim}_{t\to\infty} \xi_t = \infty \tag{29.10}$$

(make a picture of a function defined for $t \in [0,\,\infty)$ and oscillating wider and wider as $t \to \infty$).

So almost surely $\tau_{(a,\,\infty)} < \infty$, $\tau_{(-\infty,\,b)} < \infty$. What about the expectations of these random variables?

From (29.7), using the monotone-convergence theorem, we get:

$$E_x\tau_{(a,\,\infty)} = \lim_{b\to\infty} E_x\tau_{(a,\,b)} = \lim_{b\to\infty} (b-x)(x-a) = \infty. \tag{29.11}$$

When we started, in the elementary probability theory course, with expectations, we were shown some examples of random variables having no finite expectations; these examples seemed to be pretty artificial. But here we have quite a natural example of a random variable $\tau = \tau_{(a,\,\infty)}$ having infinite expectation.

In fact, we have already established the facts that $\tau_{(a,\infty)}$ is finite almost surely and that $E_x\tau_{(a,\infty)}$: we wrote explicitly the distribution of this random variable (see Lecture 19). I showed this other method because it can be applied to a wide class of other one-dimensional diffusion processes.

Example 29.2. Now let us go to the d-dimensional Wiener process, $d > 1$. Let G be the sphere of radius R centered at the origin: $G = \{\boldsymbol{x}\colon |\boldsymbol{x}| < R\}$ (in the two-dimensional case, the circle; but if the dimensionality is not specified, let us speak of it as a *sphere*). Let us find $m(\boldsymbol{x}) = E_{\boldsymbol{x}}\tau_G$.

We should be looking for it as the solution of the Dirichlet problem

$$\begin{aligned} \tfrac{1}{2}\Delta m(\boldsymbol{x}) &= -1, \qquad \boldsymbol{x} \in G, \\ m(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in \partial G. \end{aligned} \tag{29.12}$$

In the one-dimensional case, the solution was a quadratic function; it turns out that in the multidimensional case too

$$m(\boldsymbol{x}) = d^{-1}(R^2 - |\boldsymbol{x}|^2) = d^{-1}\bigl(R^2 - (x_1^2 + \ldots + x_d^2)\bigr), \qquad |\boldsymbol{x}| \le R. \tag{29.13}$$

Indeed, inside the sphere

$$\tfrac{1}{2}\Delta m(\boldsymbol{x}) = \tfrac{1}{2}\sum_{i=1}^{d}\frac{\partial^2 m}{\partial x_i^2} = \tfrac{1}{2}\sum_{i=1}^{d}(-2) = -1; \tag{29.14}$$

and the boundary condition is, of course, satisfied.

Can the function $m(\boldsymbol{x})$ be extended outside the region and its boundary as... etc, etc? In fact, it is done by means of the same formula $R^2 - |\boldsymbol{x}|^2$: this function is defined and smooth in the whole space $\mathbb{R}^d$, and it grows, with its derivatives, certainly not faster than exponentially. So we have that the expectation of τ_G is finite, and it is given by formula (29.13).

Example 29.3. Let G be the region between two spheres: $G = \{\boldsymbol{x}\colon \rho < |\boldsymbol{x}| < R\}$. Is $E_{\boldsymbol{x}}\tau_G < \infty$?

It's not difficult to find the solution of the Dirichlet problem $\frac{1}{2}\Delta m(\boldsymbol{x}) = -1$, $\boldsymbol{x} \in G$, $m(\boldsymbol{x}) = 0$, $\boldsymbol{x} \in \partial G$; but we don't need it: certainly the time τ_G of leaving the region between the two spheres is not greater than the time of leaving the greater sphere – and the expectation of that time is finite by the previous example.

In general, the time τ_G of leaving an arbitrary *bounded* region G is finite.

Example 29.4. For the region $G = G_{\rho R}$ of the previous example, what is the probability of the Wiener process $\boldsymbol{\xi}_t$ leaving it through the smaller sphere:

$$u(\boldsymbol{x}) = P_{\boldsymbol{x}}\{|\boldsymbol{\xi}_{\tau_G}| = \rho\} \tag{29.15}$$

(and what is the probability of leaving G through the R-sphere)?

We have to solve the Dirichlet problem

$$\begin{aligned} \tfrac{1}{2}\,\Delta u(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in G_{\rho R}, \\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial G_{\rho R}, \end{aligned} \tag{29.16}$$

where

$$\varphi(\boldsymbol{x}) = \begin{cases} 1, & |\boldsymbol{x}| = \rho, \\ 0, & |\boldsymbol{x}| = R. \end{cases} \tag{29.17}$$

It is easily checked (by straightforward differentiation) that the solution is

$$u(\boldsymbol{x}) = u_{\rho R}(\boldsymbol{x}) = \frac{\ln R - \ln |\boldsymbol{x}|}{\ln R - \ln \rho}, \qquad \rho \le |\boldsymbol{x}| \le R, \tag{29.18}$$

for the dimension $d = 2$, and

$$u(\boldsymbol{x}) = u_{\rho R}(\boldsymbol{x}) = \frac{|\boldsymbol{x}|^{2-d} - R^{2-d}}{\rho^{2-d} - R^{2-d}}, \qquad \rho \le |\boldsymbol{x}| \le R, \tag{29.19}$$

for $d > 2$.

Again the formulas (29.18), (29.19) define a function beyond the confines of the region G; but these formulas don't define a function that is smooth in the whole space: the limit of these expressions as $\boldsymbol{x} \to \boldsymbol{0}$ is $+\infty$. Of course we can change this function outside G so that the extension is smooth; for example like this. We construct a smooth (twice continuously differentiable function $h(x)$ so that $h(x) = 1$ for $x \le 0$ and $h(x) = 0$ for $x \ge 1$, e. g.,

$$h(x) = \begin{cases} 1, & x \le 0, \\ 1 - 10x^3 + 15x^4 - 6x^5, & 0 \le x \le 1, \\ 0, & x \ge 1; \end{cases} \tag{29.20}$$

and we take

$$u(\boldsymbol{x}) = u_0(\boldsymbol{x}) \cdot h\big(2(\rho - |\boldsymbol{x}|)/\rho\big). \tag{29.21}$$

This function is equal to 0 in the sphere $\{\boldsymbol{x} : |\boldsymbol{x}| \le \rho/2\}$, and it is twice continuously differentiable everywhere.

This is enough for the present example, but in order not to have to extend our solution of the Dirichlet problem every time we come to this, let me formulate a general result in this field:

Theorem 29.1. *Let G be a region in $\mathbb{R}^d$ such that $E_{\boldsymbol{x}}\tau_G < \infty$. Let $g(\boldsymbol{x})$, $\boldsymbol{x} \in G$, and $\varphi(\boldsymbol{x})$, $\boldsymbol{x} \in \partial G$, be bounded functions. Suppose $u(\boldsymbol{x})$ is a bounded solution of the Dirichlet problem* (28.22).

Then the equality (28.23) *holds for $\boldsymbol{x} \in G \cup \partial G$.*

That is: we can do *without* the condition of the solution being extended to the whole space.

Proof. Let $G_1 \subset G_2 \subset ... \subset G_n \subset ... \subset G$ be a sequence of bounded regions such that $G_n \cup \partial G_n \subset G_{n+1}$, and $\bigcup_{n=1}^{\infty} G_n = G$ (make a picture, taking the region G unbounded). Let us change the function $u(\boldsymbol{x})$ outside $G_n \cup \partial G_n$ so that the new function is twice continuously differentiable, and equal to 0 outside the region G_{n+1} (this can be done by multiplying the function $u(\boldsymbol{x})$ by the function h taken of an appropriate argument; I don't give the details). Let $u_n(\boldsymbol{x})$ be the function $u(\boldsymbol{x})$ changed so, and let $\tau_n = \tau_{G_n}$ be the time at which the process $\boldsymbol{\xi}_t$ leaves the region G_n. By Theorem 28.6 we have:

$$u_n(\boldsymbol{x}) = E_{\boldsymbol{x}}\Bigl(u_n(\boldsymbol{\xi}_{\tau_n}) - \int_0^{\tau_n} g(\boldsymbol{\xi}_s)\, ds\Bigr). \tag{29.22}$$

Since $\boldsymbol{x} \in G_n$ (this, at least for sufficiently large n), $\boldsymbol{\xi}_s \in G_n$, and $\boldsymbol{\xi}_{\tau_n} \in G_n \cup \partial G_n$, we have:

$$u(\boldsymbol{x}) = E_{\boldsymbol{x}}\Bigl(u(\boldsymbol{\xi}_{\tau_n}) - \int_0^{\tau_n} Lu(\boldsymbol{\xi}_s)\, ds\Bigr). \tag{29.23}$$

Now, as $n \to \infty$, we have $\tau_n \to \tau$, $\boldsymbol{\xi}_{\tau_n} \to \boldsymbol{\xi}_\tau \in \partial G$, $u(\boldsymbol{\xi}_{\tau_n}) \to \varphi(\boldsymbol{\xi}_\tau)$, $\displaystyle\int_0^{\tau_n} g(\boldsymbol{\xi}_s)\, ds \to \int_0^{\tau} g(\boldsymbol{\xi}_s)\, ds$. All random variables under the expectation sign in (29.23) are dominated in absolute value by the same random variable $\|u\| + \|g\| \cdot \tau_G$, and this random variable has a finite expectation. By the dominated-convergence theorem we get (28.23).

Theorem 29.1′. *Let G be a region in $\mathbb{R}^d$ such that almost surely $\tau = \tau_G < \infty$* (finiteness of the expectation $E_{\boldsymbol{x}}\tau_G$ is not required). *Let $\varphi(\boldsymbol{x})$, $\boldsymbol{x} \in \partial G$, be a bounded function. Suppose $u(\boldsymbol{x})$ is a bounded solution of the Dirichlet problem*

$$\begin{aligned} Lu(\boldsymbol{x}) &= 0, & \boldsymbol{x} &\in G,\\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), & \boldsymbol{x} &\in \partial G. \end{aligned} \tag{29.24}$$

Then

$$u(\boldsymbol{x}) = E_{\boldsymbol{x}}\varphi(\boldsymbol{\xi}_\tau). \tag{29.25}$$

The **proof** is the same.

From Theorem 29.1′ uniqueness of a bounded solution of the Dirichlet problem follows (in the case that τ_G is almost surely finite).

Note that without the condition of the solution $u(\boldsymbol{x})$ being bounded, there is no uniqueness (and of course, formulas (28.23), (29.25) do not necessarily hold).

Now let us return to the d-dimensional Wiener process an Example 29.4.

By Theorem 29.1 (or 29.1′), the probability (29.15) is given by formulas (29.18), (29.19).

In Example 29.1 we looked at what happened when the left end a of the interval (a, b) goes to $-\infty$, or if $b \to \infty$. Let us look at what happens in the multidimensional case when $R \to \infty$ and what as $\rho \to 0^+$.

The family of events

$$A_{\rho R} = \{|\boldsymbol{\xi}_{\tau_{G_{\rho R}}}| = \rho\} \tag{29.26}$$

(the process leaves $G_{\rho R}$ through the smaller sphere, or circle) is non-decreasing as R grows, for fixed ρ:

$$A_{\rho R_1} \subseteq A_{\rho R_2} \ \text{ for } R_1 < R_2. \tag{29.27}$$

This means that if we left the region $G_{\rho R_1}$ through the smaller sphere (circle), so will we do with the region $G_{\rho R_2}$ (make a picture of the two regions $G_{\rho R_1}$ and $G_{\rho R_2}$, and a trajectory leaving $G_{\rho R_1}$ through the circle of radius ρ).

For fixed R, and ρ *de*creasing towards 0, the family of events $A_{\rho R}$ is *non-increasing*:

$$A_{\rho_1 R} \supseteq A_{\rho_2 R} \ \text{ for } \rho_2 < \rho_1. \tag{29.28}$$

This means that if we left the region $G_{\rho_2 R}$ through the smaller sphere (circle), so will we do with the region $G_{\rho_1 R}$ (again make a picture of the two regions $G_{\rho_1 R}$ and $G_{\rho_2 R}$, and a trajectory leaving $G_{\rho_2 R}$ through the circle of radius ρ_2).

So we have:

$$P_{\boldsymbol{x}}\Big(\bigcup_{R>|\boldsymbol{x}|} A_{\rho R}\Big) = \lim_{R\to\infty} P_{\boldsymbol{x}}(A_{\rho R}), \tag{29.29}$$

$$P_{\boldsymbol{x}}\Big(\bigcap_{\rho<|\boldsymbol{x}|} A_{\rho R}\Big) = \lim_{\rho\to 0+} P_{\boldsymbol{x}}(A_{\rho R}). \tag{29.30}$$

What remains to do is to understand what these unions and intersections are, and evaluating the limits of the function given by (29.18), (29.19).

Let us introduce the notation:

$$\tau_c = \min\{t\colon\ |\boldsymbol{\xi}_t| = c\} \tag{29.31}$$

(as always, taking $\tau_c = \infty$ if the process never reaches the sphere – or circle – of radius c). We know that almost surely $\tau_c < \infty$ for $c > |\boldsymbol{\xi}_0| = |\boldsymbol{x}|$, and even that $E_{\boldsymbol{x}}\tau_c < \infty$, because this is the first time of leaving the bounded region $\{\boldsymbol{x} : \ |\boldsymbol{x}| < c\}$ (see Theorem 29.1); we don't know whether $\tau_c < \infty$ for $c < |\boldsymbol{\xi}_0| = |\boldsymbol{x}|$ (this is the first time of leaving the *un*bounded region $\{\boldsymbol{x}\colon\ |\boldsymbol{x}| > c\}$). Using the notation τ_c, we can rewrite the event (29.26) as

$$A_{\rho R} = \{\tau_\rho < \tau_R\}. \tag{29.32}$$

As $R \to \infty$, we have, obviously,

$$\lim_{R\to\infty} \tau_R = \infty \tag{29.33}$$

(the Wiener process, being continuous, cannot go to infinity in a finite time). This suggests that

$$\bigcup_{R>|\boldsymbol{x}|} A_{\rho R} = \{\tau_\rho < \infty\}. \tag{29.34}$$

To prove this, we have to check that $\bigcup_{R>|\boldsymbol{x}|} A_{\rho R} \subseteq \{\tau_\rho < \infty\}$ and that $\{\tau_\rho^{\boldsymbol{x}} < \infty\} \subseteq \bigcup_{R>|\boldsymbol{x}|} A_{\rho R}$.

The first is obvious: if we have reached the sphere (circle) of radius ρ before reaching that of radius R for some R, then we have reached it at *some* time $< \infty$. The second: suppose for an $\omega \in \Omega$ we have $\tau_\rho(\omega) < \infty$; the function $|\boldsymbol{\xi}_t(\omega)|$ is continuous, and so there exists its maximum over a finite closed interval $\max\{|\boldsymbol{\xi}_t(\omega)|:\ 0 \le t \le \tau_\rho(\omega)\}$. For $R > \max\{|\boldsymbol{\xi}_t(\omega)|:\ 0 \le t \le \tau_\rho(\omega)\}$ we have $\tau_\rho(\omega) < \tau_R(\omega)$.

So

$$\lim_{R\to\infty} P_{\boldsymbol{x}}(A_{\rho R}) = \lim_{R\to\infty} u_{\rho R}(\boldsymbol{x}) = P_{\boldsymbol{x}}\{\tau_\rho < \infty\}. \tag{29.35}$$

Now to the limit, for fixed R, as $\rho \to 0^+$. Clearly we have

$$\bigcap_{\rho<|\boldsymbol{x}|} A_{\rho R} = \{\tau_0 < \tau_R\}, \tag{29.36}$$

where τ_0 is the time of reaching the set $\{\boldsymbol{x}:\ |\boldsymbol{x}| = 0\}$, that is, the point $\mathbf{0}$ (check the inclusions $\bigcap_{\rho<|\boldsymbol{x}|} A_{\rho R} \subseteq \{\tau_0 < \tau_R\}$, $\{\tau_0 < \tau_R\} \subseteq \bigcap_{\rho<|\boldsymbol{x}|} A_{\rho R}$ yourselves, making the appropriate picture). So we have:

$$P_{\boldsymbol{x}}\{\tau_0 < \tau_R\} = \lim_{\rho\to 0^+} P_{\boldsymbol{x}}(A_{\rho R}) = \lim_{\rho\to 0^+} u_{\rho R}(\boldsymbol{x}). \tag{29.37}$$

Now let us evaluate the limits (29.35), (29.37). First, as $R \to \infty$. We have, for the dimension $d = 2$:

$$P_{\boldsymbol{x}}\{\tau_\rho < \infty\} = \lim_{R\to\infty} \frac{\ln R - \ln|\boldsymbol{x}|}{\ln R - \ln\rho} = 1: \tag{29.38}$$

the process almost surely reaches every small circle of a positive radius ρ centered at 0.

If we take circles with an arbitrary center $\boldsymbol{x}_0 \in \mathbb{R}^2$ instead of those centered at the origin $\mathbf{0}$, everything remains the same. So the Wiener trajectory almost surely reaches *every* circle: the trajectory $\boldsymbol{\xi}_t$, $t \in [0, \infty)$, covers a dense subset of the plane $\mathbb{R}^2$.

For the dimension $d > 2$:

$$P_{\boldsymbol{x}}\{\tau_\rho < \infty\} = \lim_{R\to\infty} \frac{|\boldsymbol{x}|^{2-r} - R^{2-r}}{\rho^{2-r} - R^{2-r}} = \Big(\frac{\rho}{|\boldsymbol{x}|}\Big)^{r-2}. \tag{29.39}$$

We see that a sphere is reached from the outside with some positive probability that is less than 1.

Now about the limits as $\rho \to 0$. We have, both in the cases $d = 2$ and $d > 2$, for $\boldsymbol{x} \ne 0$:

$$P_{\boldsymbol{x}}\{\tau_0 < \tau_R\} = \lim_{\rho\to 0^+} P_{\boldsymbol{x}}\{\tau_\rho < \tau_R\} = \lim_{\rho\to 0^+} u_{\rho R}(\boldsymbol{x}) = 0. \tag{29.40}$$

That is, if we do not start from the point $\mathbf{0}$, almost surely we never reach this point before going out of the sphere (circle) of radius R. Since this is true for *every* $R > |\boldsymbol{x}|$, almost surely we *never* reach $\mathbf{0}$ (note the contrast with the one-dimensional case, where almost surely we reach *every* point of the line).

If we start from the point $\boldsymbol{\xi}_0 = \mathbf{0}$ at time $t = 0$, we leave this point (because for every $t > 0$ the random vector $\boldsymbol{\xi}_t$ has a normal distribution – a continuous one, and $P_{\mathbf{0}}\{\boldsymbol{\xi}_t = \mathbf{0}\} = 0$), and never return to it.

If, instead of considering spheres (circles) centered at $\mathbf{0}$, we take them centered at an arbitrary point $\boldsymbol{x}_0$ (and replace $|\boldsymbol{x}|$ with $|\boldsymbol{x} - \boldsymbol{x}_0|$), we get that for every $\boldsymbol{x}_0 \ne \boldsymbol{x}$

$$P_{\boldsymbol{x}}\{\boldsymbol{\xi}_t \text{ ever reaches } \boldsymbol{x}_0\} = 0: \tag{29.41}$$

for every point $\boldsymbol{x}_0$ different from our starting point almost surely we never reach it.

Does this mean that almost surely the process $\boldsymbol{\xi}_t$ never reaches any point $\boldsymbol{x}_0$ different from $\boldsymbol{x}$ (which seems absurd)?

Let us introduce the event

$$B_{\boldsymbol{x}_0} = \{\boldsymbol{\xi}_t \text{ reaches } \boldsymbol{x}_0 \text{ at some time}\}. \tag{29.42}$$

The question is: we know that $P_{\boldsymbol{x}}(B_{\boldsymbol{x}_0}) = 0$ for every $\boldsymbol{x}_0 \neq \boldsymbol{x}$; does it follow that

$$P_{\boldsymbol{x}}\{\boldsymbol{\xi}_t \text{ ever reaches any point } \boldsymbol{x}_0 \neq \boldsymbol{x}\} = P_{\boldsymbol{x}}\Big(\bigcup_{\boldsymbol{x}_0 \neq \boldsymbol{x}} B_{\boldsymbol{x}_0}\Big) = 0? \tag{29.43}$$

If the union $\bigcup_{\boldsymbol{x}_0 \neq \boldsymbol{x}} B_{\boldsymbol{x}_0}$ were a *countable* one, it would follow (by the countable additivity of the probability measure); but the set of all points $\boldsymbol{x}_0 \in \mathbb{R}^d$ that are different from $\boldsymbol{x}$ is clearly *un*countable: so we cannot get (29.43) this way.

But maybe (29.43) still is true for some other reason? It turns out that $P_{\boldsymbol{x}}\big(\bigcup_{\boldsymbol{x}_0 \neq \boldsymbol{x}} B_{\boldsymbol{x}_0}\big) = P_{\boldsymbol{x}}\{\boldsymbol{\xi}_t$ ever reaches any point $\boldsymbol{x}_0 \neq \boldsymbol{x}\} = 1$: because, let me repeat again, at every time $t > 0$ the process $\boldsymbol{\xi}_t$ almost surely *is* at some point different from $\boldsymbol{x}$ ($P_{\boldsymbol{x}}\{\boldsymbol{\xi}_t \neq \boldsymbol{x}\} = 1$).

The next small piece was not given in Lecture 29, but I am including it in the lecture note because it fits naturally here:

We have let $R \to \infty$ for fixed ρ, obtaining the result that a two-dimensional Wiener process almost surely reaches the "sphere" (circle in fact) of every positive radius ρ, while in the case of greater dimension this is not so; and we have taken $\rho \to 0^+$ for fixed R, which told us that we almost surely cannot reach a point $\boldsymbol{x}_0 \neq \boldsymbol{x}$; what if we let first $R \to \infty$, and after that $\rho \to 0$? If we let $\rho \to 0^+$ in (29.39), we get that

$$\begin{aligned} &P_{\boldsymbol{x}}\Big(\bigcap_{\rho < |\boldsymbol{x}|} \{\tau_\rho < \infty\}\Big) \\ &= P_{\boldsymbol{x}}\{\text{the process } \boldsymbol{\xi}_t \text{ reaches every sphere of positive radius with center at } \mathbf{0}\} \\ &= \lim_{\rho \to 0^+} \Big(\frac{\rho}{|\boldsymbol{x}|}\Big)^{d-2} = 0. \end{aligned} \tag{29.44}$$

Of course, also for $\boldsymbol{x}_0 \neq \mathbf{0}$ we have: $P_{\boldsymbol{x}}\{$the process $\boldsymbol{\xi}_t$ reaches every sphere of positive radius with center at $\boldsymbol{x}_0\} = 0$. This means that in dimensions $d > 2$ the trajectory $\boldsymbol{\xi}_t$ does *not* cover a set that is dense in $\mathbb{R}^d$ (in contrast with the case of $d = 2$).