Lecture 32. The general definition of stochastic integrals. Stochastic differentials.

Let us consider two $\mathbf{L}^2$-spaces: $\mathbf{L}^2([a, b] \times \Omega, \mathcal{P}rog, \lambda_1 \times P)$ consisting of random functions; and $\mathbf{L}^2(\Omega, \mathcal{F}_b, P)$ consisting of random variables. The stochastic integral from a to b of progressively measurable square-integrable step functions defines a mapping $I:\ f(t, \omega) \mapsto \int_a^b f(t, \omega)\, dW_t$ of a subset of $\mathbf{L}^2([a, b] \times \Omega, \mathcal{P}rog, \lambda_1 \times P)$ into the space $\mathbf{L}^2(\Omega, \mathcal{F}_b, P)$ (that the stochastic integral $I(f)$ belongs to $\mathbf{L}^2(\Omega, \mathcal{F}_b, P)$ for a step random function $f \in \mathbf{L}^2([a, b] \times \Omega, \mathcal{P}rog, \lambda_1 \times P)$ follows from Theorems 31.6, 31.7). The mapping I is, according to (31.30), *isometric*; and it can be extended by continuity to the closure in $\mathbf{L}^2([a, b] \times \Omega, \mathcal{P}rog, \lambda_1 \times P)$ of its original domain of definition (consisting of progressively measurable square-integrable step random functions). And we take the random variable $I(f)$ as the definition of the stochastic integral $\int_a^b f(t, \omega)\, dW_t$.

In more concrete terms, if there is a sequence of progressively measurable square-integrable step random functions $f_n(t, \omega)$ such that $\int_a^b E\big(f_n(t, \omega) - f(t, \omega)\big)^2\, dt \to 0$, then there exists $\operatorname{l.i.m.}_{n\to\infty} \int_a^b f_n(t, \omega)\, dW_t$, and this mean-square limit is taken as the definition of $\int_a^b f(t, \omega)\, dW_t$.

Since a mean-square limit is defined only *almost* uniquely, the stochastic integral is defined also not in a unique way; but all versions of the same stochastic integral are equal to one another almost surely.

Theorem 32.1. *If a random function $f \in \mathbf{L}^2([a, b] \times \Omega, \mathcal{P}rog, \lambda_1 \times P)$ is continuous in the mean squares*: $\lim_{t' \to t} E[f(t', \omega) - f(t, \omega)]^2 \to 0$ *as* $t' \to t \in [a, b]$, *the integral* $\int_a^b f(t, \omega)\, dW_t$ *is defined and $\mathcal{F}_b$-measurable.*

Proof. For a partition $\mathfrak{T}$ of the interval $[a, b]$: $0 = t_0 < t_1 < \ldots < t_n = b$ we take $f_{\mathfrak{T}}(t, \omega) = \sum_{i=1}^n f(t_{i-1}, \omega) \cdot I_{[t_{i-1},\, t_i)}(t)$. The function $f_{\mathfrak{T}}(t, \omega)$ is a progressively measurable (adapted) step random function; and because of the mean-square continuity of f, and its *uniform* mean-square continuity (because the interval $[a, b]$ is compact), we have $\lim_{\max_{1 \le i \le n}(t_i - t_{i-1}) \to 0} \int_a^b E\big(f_{\mathfrak{T}}(t, \omega) - f(t, \omega)\big)^2\, dt = 0$, so the stochastic integral *is* defined.

Theorem 32.2. *If a random function $f \in \mathbf{L}^2([a, b] \times \Omega, \mathcal{P}rog, \lambda_1 \times P)$ is continuous in the mean squares except at finitely many points in $[a, b]$ and has one-sided mean-square limits at these exceptional points, then the stochastic integral is defined.*

The **proof** is essentially the same except we have to take care of the neighborhoods of the exceptional points – just the same way as we prove that a non-random piecewise continuous function with one-sided limits at the exceptional points id Riemann integrable.

In particular, Theorem 32.2 can be applied to progressively measurable step random functions – only we don't need any theorems for that.

With a little more effort we can prove that the stochastic integral exists for *every* random function belonging to $\mathbf{L}^2([a,\, b] \times \Omega,\, \mathcal{P}rog,\, \lambda_1 \times P)$: we approximate a random function f with functions $g_{\mathfrak{T}}(t) = \sum_{i=2}^{n} \Big(\int_{t_{i-2}}^{t_{i-1}} f(s,\, \omega)\, ds \Big/ (t_{i-1} - t_{i-2}) \Big) \cdot I_{[t_{i-1},\, t_i)}(t)$.

Quite obviously, stochastic integrals of general progressively measurable random functions are linear with respect to their integrands: the equalities (31.21), (31.22) are satisfied; also $\displaystyle\int_a^c = \int_a^b + \int_b^c$. But in contrast with the case of step random functions, for which these formulas hold for *all* ω, in the general case they hold only *almost surely*. Since the stochastic integrals are defined by limit passage from integrals of step functions, these properties are proved just by this limit passage; and since mean-square limits are defined uniquely only almost everywhere, this is how the equalities hold. Also by limit passage from formulas (31.28) – (31.30) for step random functions we get that for a general square-integrable progressively measurable random function $f(t,\, \omega)$ we have:

$$E \int_a^b f(t,\, \omega)\, dW_t = 0, \tag{32.1}$$

$$E\Big(\int_a^b f(t,\, \omega)\, dW_t \Big\| \mathcal{F}_a \Big) = 0, \tag{32.2}$$

$$E\Big(\int_a^b f(t,\, \omega)\, dW_t \Big)^2 = E \int_a^b f(t,\, \omega)^2\, dt = \int_a^b E f(t,\, \omega)^2\, dt. \tag{32.3}$$

The second remark: We defined, in Theorem 32.1, the stochastic integral $\displaystyle\int_a^b f(t,\, \omega)\, dW_t$ of a mean-square continuous random function as the limit in mean of the integrals $\displaystyle\int_a^b f_{\mathfrak{T}}(t,\, \omega)\, dW_t$ of step random functions $f_{\mathfrak{T}}(t,\, \omega) = \sum_{i=1}^{n} f(t_{i-1},\, \omega) \cdot I_{[t_{i-1},\, t_i)}(t)$. This definition can, of course, be written in the form

$$\int_a^b f(t,\, \omega)\, dW_t = \underset{\max_{1 \le t \le n}(t_i - t_{i-1}) \to 0}{\text{l.i.m.}} \sum_{i=1}^{n} f(t_{i-1},\, \omega) \cdot (W_{t_i} - W_{t_{i-1}}). \tag{32.4}$$

That is, in the Riemann-Stieltjes sums we take the integrand at the left end t_{i-1} of every interval $[t_{i-1},\, t_i]$ of the partition. Can an arbitrary other point t_i^* be chosen in the interval from t_{i-1} to t_i to take the value $f(t_i^*,\, \omega)$ of the random function at it? No, in general, if we want our proof of mean-square convergence to stand. Indeed, if we take, for example, in lieu of $f_{\mathfrak{T}}(t,\, \omega)$,

$$f^{\mathfrak{T}}(t,\, \omega) = \sum_{i=1}^{n} f(t_i,\, \omega) \cdot I_{[t_{i-1},\, t_i)}(t), \tag{32.5}$$

this step function won't be, in general, adapted to the family of σ-algebras $\mathcal{F}_t$: its value, say, at the point $(t_i + t_{i-1})/2$ is equal to $f(t_i, \omega)$, which may be measurable with respect to $\mathcal{F}_{t_i}$, but not with respect to $\mathcal{F}_{(t_i+t_{i-1})/2}$.

So for general progressively measurable random functions $f(t, \omega)$ the definition

$$\int_a^b f(t, \omega)\, dW_t = \underset{\max_{1\le i\le n}(t_i - t_{i-1})\to 0}{\text{l.i.m.}} \sum_{i=1}^n f(t_i^*, \omega)\cdot(W_{t_i} - W_{t_{i-1}}) \tag{32.6}$$

with arbitrary $t_i^* \in [t_{i-1}, t_i]$ is impossible. To be precise, *our method* of proving that the mean-square limit exists is not valid for the limit (32.6) for general progressively measurable integrands.

Example 32.1. The stochastic integral $\int_a^b W_t\, dW_t$ exists; let us try to find an expression for it.

We have, according to (32.4):

$$\int_a^b W_t\, dW_t = \underset{\max_{1\le i\le n}(t_i - t_{i-1})\to 0}{\text{l.i.m.}} \sum_{i=1}^n W_{t_{i-1}}\cdot(W_{t_i} - W_{t_{i-1}}). \tag{32.7}$$

The factor $W_{t_{i-1}}$ can be represented as $W_a + \sum_{j=1}^{i-1}(W_{t_j} - W_{t_{j-1}})$; so the sum in (32.7) can be rewritten as

$$\begin{aligned} W_a\cdot\sum_{i=1}^n(W_{t_i} - W_{t_{i-1}}) &+ \sum_{1\le j<i\le n}(W_{t_j} - W_{t_{j-1}})\cdot(W_{t_i} - W_{t_{i-1}}) \\ &= W_a\cdot(W_b - W_a) + \sum_{1\le j<i\le n}(W_{t_j} - W_{t_{j-1}})\cdot(W_{t_i} - W_{t_{i-1}}). \end{aligned} \tag{32.8}$$

But almost the same terms will be in the following expression:

$$\begin{aligned} (W_b - W_a)^2 &= \Big(\sum_{i=1}^n(W_{t_i} - W_{t_{i-1}})\Big)^2 \\ &= \sum_{i=1}^n(W_{t_i} - W_{t_{i-1}})^2 + 2\sum_{1\le j<i\le n}(W_{t_j} - W_{t_{j-1}})\cdot(W_{t_i} - W_{t_{i-1}}). \end{aligned} \tag{32.9}$$

So we have:

$$\begin{aligned} \sum_{i=1}^n W_{t_{i-1}}\cdot(W_{t_i} - W_{t_{i-1}}) &= W_a\cdot(W_b - W_a) + \frac{1}{2}\Big[(W_b - W_a)^2 - \sum_{i=1}^n(W_{t_i} - W_{t_{i-1}})^2\Big] \\ &= \frac{1}{2}\Big[W_b^2 - W_a^2 - \sum_{i=1}^n(W_{t_i} - W_{t_{i-1}})^2\Big]. \end{aligned} \tag{32.10}$$

According to (30.15), the mean-square limit of the sum is equal to $b - a$, so

$$\int_a^b W_t\, dW_t = \frac{1}{2}\,[W_b^2 - W_a^2 - (b - a)]. \tag{32.11}$$

It is easy to check that

$$\underset{\max_{1\le i\le n}(t_i-t_{i-1})\to 0}{\text{l.i.m.}} \sum_{i=1}^{n} W_{t_i}\cdot(W_{t_i}-W_{t_{i-1}}) = \frac{1}{2}\,[W_b^2 - W_a^2 + (b-a)] \tag{32.12}$$

(consider the difference of the sums with W_{t_i} and with $W_{t_{i-1}}$); so the limit depends on the choice of the points $t_i^* \in [t_{i-1},\, t_i]$, and the limit (32.6) for *all* choices of t_i^* does not exist.

In considering stochastic integral equations, we'll have to consider stochastic integrals $\eta_t = \eta_t(\omega) = \int_{t_0}^{t} f(s,\,\omega)\;dW_s$ with variable upper limit $t \ge t_0$, being random functions.

Theorem 32.3. *The random function* $\eta_t = \int_a^t f(s,\,\omega)\;dW_s$, $t\ge a$, *is a martingale.*

Proof. That η_t is adapted to our family of σ-algebras is obvious; and for $t<u$

$$E\Big(\int_a^u f(s,\,\omega)\;dW_s - \int_a^t f(s,\,\omega)\;dW_s\Big\|\mathcal{F}_t\Big) = E\Big(\int_t^u f(s,\,\omega)\;dW_s\Big\|\mathcal{F}_t\Big) = 0 \tag{32.13}$$

by (32.2).

The random function η_t is mean-square continuous (easy to prove); but what are the properties of its *sample functions* $\eta_t(\omega)$? Can we state that necessarily almost all sample functions are continuous?

In general, this question makes no sense: we can only state that there exist *versions* of stochastic integrals such that $\eta_t(\omega)$ is continuous:

Theorem 32.4. *For a random function* $f(t,\,\omega)$, $a\le t\le b$, *of the kind considered above*, there exists *such a version of the stochastic integral*

$$\eta_t = \int_a^t f(s,\,\omega)\;dW_s, \qquad t\in[a,\,b], \tag{32.14}$$

that almost surely its sample function $\eta_t(\omega)$ *is continuous for* $t\in[a,\,b]$ *and that* η_t *is progressively measurable.*

Proof. We can choose a sequence of progressively measurable step functions $f_n(t,\,\omega)$ so that $\int_a^b E[f_n(s,\,\omega)-f(s,\,\omega)]^2\,ds \to 0$ $(n\to\infty)$. Of course, we can choose them so that this integral is $<1/10^n$. It follows from this that

$$\begin{aligned}\int_a^b & E[f_{n+1}(s,\,\omega)-f_n(s,\,\omega)]^2\,ds \\ &\le 2\int_a^b E[f_{n+1}(s,\,\omega)-f(s,\,\omega)]^2\,ds + \int_a^b E[f_n(s,\,\omega)-f(s,\,\omega)]^2\,ds \\ &< 2/10^n + 2/10^{n+1} < 3/10^n.\end{aligned} \tag{32.15}$$

The stochastic integral $\int_a^t [f_{n+1}(s, \omega) - f_n(s, \omega)]\, dW_s$ of a *step* random function is continuous in t by Theorem 31.3, and it is a martingale by Theorem 31.8. By the Kolmogorov inequality (Theorem 28.3 applied to the submartingale $\Big(\int_a^t [f_{n+1}(s, \omega) - f_n(s, \omega)]\, dW_s\Big)^2$) we have:

$$P\Big\{\max_{a \le t \le b}\Big|\int_a^t [f_{n+1}(s, \omega) - f_n(s, \omega)]\, dW_s\Big| \ge 1/2^n\Big\}$$
$$\le \frac{\displaystyle\int_a^b E[f_{n+1}(s, \omega) - f_n(s, \omega)]^2\, ds}{(1/2^n)^2} < 3/2^n. \tag{32.16}$$

The series of these probabilities converges, so almost surely there will occur only finitely many of the events, and for sufficiently large n the difference of the stochastic integrals is $< 1/2^n$. Again, the series $\sum_{n=1}^{\infty} 1/2^n$ converges, so the series

$$\int_a^t f_1(s, \omega)\, dW_s + \sum_{n=1}^{\infty} \int_a^t [f_{n+1}(s, \omega) - f_n(s, \omega)]\, dW_s \tag{32.17}$$

converges almost surely uniformly over $t \in [a, b]$; and the sum of a uniformly convergent series of continuous functions is continuous. The series (32.17) is the almost-sure limit of $\int_a^t f_n(s, \omega)\, dW_s$; and it is a version of the mean-square limit having almost surely continuous sample functions.

Now about the progressive measurability. As the version of the stochastic integral we can take

$$\eta_t(\omega) = \begin{cases} \lim\limits_{n\to\infty} \displaystyle\int_a^t f_n(s, \omega)\, dW_s & \text{if a finite limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{32.18}$$

The finite limit here exists almost surely, so $\eta_t(\omega)$ is a version of the mean-square limit. And this random function is progressively measurable as a function made up of a limit of a sequence of progressively measurable random functions on the set where the finite limit exists, which belongs to the σ-algebra $\mathcal{P}rog$; and of another measurable function: a constant, on the complement of this set.

In comprehensive books on stochastic integrals, the existence of the stochastic integral is proved under a weaker condition instead of

$$E\int_a^b f(t, \omega)^2\, dt < \infty: \tag{32.19}$$

only that

$$\int_a^b f(t, \omega)^2\, dt < \infty \qquad \text{almost surely.} \tag{32.20}$$

We are not going to consider stochastic integrals of random functions satisfying only the condition (32.20), because, while under the condition (32.19) the stochastic integral has zero expectation:

$$E\int_a^b f(t, \omega)\, dW_t = 0 \tag{32.21}$$

(because the expectation of the limit is equal to the limit of the expectation if the limit of random variables is understood as a mean-square limit), under the condition (32.20) it is no longer necessarily so.

So in all stochastic integrals to come up in this course, the condition (32.19) will be always assumed.

Now I want to introduce a concept I have not touched upon before now: that of *stochastic differentials*.

Let W_t, $t \geq t_0$, be a Wiener process; let ξ_t, $t \geq t_0$, be another stochastic process. We'll say that the process ξ_t *has a stochastic differential* if its trajectories are almost surely continuous, and there exist two progressively measurable functions $f(t, \omega)$ and $g(t, \omega)$, $t \geq t_0$, such that for every $t \geq t_0$ almost surely

$$\xi_t = \xi_{t_0} + \int_{t_0}^{t} f(s, \omega)\ ds + \int_{t_0}^{t} g(s, \omega)\ dW_s. \tag{32.22}$$

(The first integral is understood as a Lebesgue one, the integration being performed separately for (almost) every $\omega \in \Omega$, and the second one as a stochastic integral.) In this case we write the stochastic differential of the process ξ_t as

$$d\xi_t = f(t, \omega)\, dt + g(t, \omega)\, dW_t. \tag{32.23}$$

In the case of an r-dimensional Wiener process $\boldsymbol{W}_t = (W_t^1, ..., W_t^n)$, the stochastic differential of a process ξ_t is written as

$$d\xi_t = f(t, \omega)\, dt + \sum_{k=1}^{r} g_k(t, \omega)\ dW_t^k, \tag{32.24}$$

and this means, by definition, that almost surely

$$\xi_t = \xi_{t_0} + \int_{t_0}^{t} f(s, \omega)\ ds + \sum_{k=1}^{r} \int_{t_0}^{t} g_k(s, \omega)\ dW_s^k. \tag{32.25}$$

In particular, it may be that every coordinate ξ_t^i of a d-dimensional stochastic process $\boldsymbol{\xi}_t$ has a stochastic differential:

$$d\xi_t^i = f_i(t, \omega)\, dt + \sum_{k=1}^{r} g_{ik}(t, \omega)\ dW_t^k, \qquad 1 \leq i \leq d, \tag{32.26}$$

where $\boldsymbol{f}(t, \omega) = \big(f_1(t, \omega), ..., f_d(t, \omega)\big)$ is a vector-valued random function, and $G(t, \omega) = \big(g_{ik}(t, \omega)\big)$ a matrix-valued one. If we write all vectors as column vectors rather than row ones, the natural notation for (32.26) is

$$d\boldsymbol{\xi}_t = \boldsymbol{f}(t, \omega)\, dt + G(t, \omega)\ d\boldsymbol{W}_t. \tag{32.27}$$

Example 32.1 shows us that $W_t^2 = W_{t_0}^2 + 2\int_{t_0}^{t} W_s\, dW_s + \int_{t_0}^{t} 1\ ds$; in the language of stochastic differentials it is rewritten as

$$d(W_t^2) = 2W_t\, dW_t + 1\, dt. \tag{32.28}$$

We'll be able to get more examples of stochastic differentials of random functions when we know what is called Itô's formula; but about this, later. In the next lecture we'll consider stochastic differential (in fact, integral) equations.