MATH 777 MARKOV STOCHASTIC PROCESSES SPRING 2009

Lecture 34. Itô's formula, end of the proof. Diffusion processes.

Before we finish the proof of Theorem 33.1, let us get some examples of applications of Itô's formula.

First of all, for $\xi_t = W_t$, $F(t,\, x) = x^2$ we have: $\dfrac{\partial F}{\partial t} = 0$, $\dfrac{\partial F}{\partial x} = 2x$, $\dfrac{\partial^2 F}{\partial x^2} = 2$,

$$dW_t^2 = dt + 2W_t\, dW_t: \tag{34.1}$$

we recognize formula (32.28).

Now *you* provide examples by solving problems:

$\boxed{\textbf{62}}$ Let $\eta_t = e^{-\alpha t} e^{\sigma W_t}$, where W_t is a Wiener process. For a given constant $\sigma \neq 0$, can we choose the coefficient α so that η_t is a martingale?

$\boxed{\textbf{63}}$ Let the stochastic process ξ_t, $t \geq 0$, be defined as $\xi_t = \sin W_t$, where W_t is a Wiener process starting at $t = 0$ from a point $x_0 \in \mathbb{R}^1$ (i. e., $W_0 = x_0$). This process takes only values in the interval $[-1,\, 1]$.

Prove or disprove that ξ_t is a solution of the stochastic differential equation

$$d\xi_t = \sqrt{1 - \xi_t^2}\, dW_t - \tfrac{1}{2}\, \xi_t\, dt. \tag{34.2}$$

Now we return to the **proof of Theorem 33.1.**

We have already considered a smaller partition $\mathfrak{S}$ within a larger, $\mathfrak{T}$ (even if they were partitions of different time intervals, $\mathfrak{S}$ of only a small interval $[c,\, d]$). Now we are going to consider two partitions of *the same interval* $[c,\, d]$: a larger one, $\mathfrak{U}$ (capital Gothic "U"), with partition points $u_0 = c < u_1 < u_2 < \dots < u_m = d$, and a smaller one, $\mathfrak{S}$, with all partition points u_j of the larger partition included in the smaller one: $u_j = s_{i_j}$ (draw a picture). We have:

$$\begin{gathered} c = u_0 = s_0 < s_1 < \dots < s_{i_1} = u_1 < s_{i_1+1} < \dots < s_{i_2-1} < s_{i_2} = u_2 < s_{i_2+1} < \dots \\ \dots < s_{i_m-1} < s_{i_m} = u_m = d. \end{gathered} \tag{34.3}$$

Let us take together the summands in the sum (33.38) corresponding to the same interval from u_{j-1} to u_j. Then we can write the difference of both sides in (33.38) as

$$\sum_{j=1}^{m} \Big[\sum_{i=i_{j-1}+1}^{i_j} \frac{\zeta^2}{2} \cdot \frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) \cdot (W_{s_i} - W_{s_{i-1}})^2 - \int_{u_{j-1}}^{u_j} \frac{\zeta^2}{2} \cdot \frac{\partial^2 F}{\partial x^2}(s,\, \xi_s)\, ds \Big]. \tag{34.4}$$

If we change in this formula the arguments in $\dfrac{\partial^2 F}{\partial x^2}$ to u_{j-1}, $\xi_{u_{j-1}}$, we obtain

$$\sum_{j=1}^{m} [\sum_{i=i_{j-1}+1}^{i_j} \frac{\zeta^2}{2} \cdot \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}}) \cdot (W_{s_i} - W_{s_{i-1}})^2 - \frac{\zeta^2}{2} \cdot \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}}) \cdot (u_j - u_{j-1})], \tag{34.5}$$

and the $\| \ \|_2$-norm of this random variable is less or equal than

$$\sum_{j=1}^{m} \frac{C^2 \cdot \big(\sup\big|\frac{\partial^2 F}{\partial x^2}\big|\big)^2}{2} \cdot \sqrt{E\Big(\sum_{i=i_{j-1}+1}^{i_j} (W_{s_i} - W_{s_{i-1}})^2 - (u_j - u_{j-1})\Big)^2}. \tag{34.6}$$

We have calculated the expectation under the square root sign: it is equal to $2 \sum_{i=i_{j-1}+1}^{i_j} (s_i - s_{i-1})^2$ (see formula (30.15)). This sum goes to 0 as the partition $\mathfrak{S}$ becomes infinitely small, and the sum (34.5) converges to 0 in the mean squares.

To get that (34.4) also converges to 0, we have to prove that the differences between the corresponding terms in (34.4) and in (34.5):

$$\sum_{j=1}^{m} \sum_{i=i_{j-1}+1}^{i_j} \frac{\zeta^2}{2} \cdot \big[\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\big] \cdot (W_{s_i} - W_{s_{i-1}})^2, \tag{34.7}$$

$$\sum_{j=1}^{m} \int_{u_{j-1}}^{u_j} \frac{\zeta^2}{2} \cdot \big[\frac{\partial^2 F}{\partial x^2}(s,\, \xi_s) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\big]\, ds \tag{34.8}$$

converge to 0 almost surely, or in the mean squares.

The sum (34.8) is easy. We choose a $\delta > 0$ so that $|s - s'| < \delta,\ |x - y| < \delta \ \Rightarrow\ \big|\frac{\partial^2 F}{\partial x^2}(s,\, x) - \frac{\partial^2 F}{\partial x^2}(s',\, y)\big| < \varepsilon$. For almost all $\omega \in \Omega$, there exists a positive $\delta' \le \delta$ such that $|s - s'| < \delta' \ \Rightarrow\ |\xi_s(\omega) - \xi_{s'}(\omega)| < \delta$; and for such ω, if $\max_{1 \le i \le n}(s_i - s_{i-1}) < \delta'$, we have:

$$\Big|\sum_{j=1}^{m} \int_{u_{j-1}}^{u_j} \frac{\zeta^2}{2} \cdot \big[\frac{\partial^2 F}{\partial x^2}(s,\, \xi_s) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\big]\, ds\Big| \le \frac{C^2}{2} \cdot \varepsilon. \tag{34.9}$$

This means that the sum (34.8) converges to 0 almost surely.

For (34.7), we prove mean-square convergence. We have:

$$\begin{aligned}
&\Big\|\sum_{j=1}^{m} \sum_{i=i_{j-1}+1}^{i_j} \frac{\zeta^2}{2} \cdot \big[\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \zeta_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \zeta_{u_{j-1}})\big] \cdot (W_{s_i} - W_{s_{i-1}})^2\Big\|_2 \\
&\quad\le \frac{C^2}{2} \sum_{j=1}^{m} \sum_{i=i_{j-1}+1}^{i_j} \Big\|\big(\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\big) \cdot (W_{s_i} - W_{s_{i-1}})^2\Big\|_2 \\
&\quad= \frac{C^2}{2} \sum_{j=1}^{m} \sum_{i=i_{j-1}+1}^{i_j} \sqrt{E\Big(\big(\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\big)^2 \cdot (W_{s_i} - W_{s_{i-1}})^4\Big)}.
\end{aligned} \tag{34.10}$$

By Schwarz's inequality, the expectation under the square root sign is less or equal than

$$\sqrt{E\big(\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\big)^4} \cdot \sqrt{E(W_{s_i} - W_{s_{i-1}})^8}. \tag{34.11}$$

The expectation of the eighth power of a normal random variable with parameters $(0,\, s_i - s_{i-1})$

$$E(W_{s_i} - W_{s_{i-1}})^8 = 1 \cdot 3 \cdot 5 \cdot 7 \cdot (s_i - s_{i-1})^4. \tag{34.12}$$

As for $E\big(\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\big)^4$, we estimate it just as in the proof of (33.35) (see formula (33.47)): for $\max_{1 \le j \le m}(u_j - u_{j-1})$ less than some positive δ' we have:

$$E\Big(\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\Big)^4 \le \varepsilon \cdot \big(2 \sup\big|\frac{\partial^2 F}{\partial x^2}\big|\big)^4 + \varepsilon^4. \tag{34.13}$$

So for such partitions $\mathfrak{U}$, and partitions $\mathfrak{S}$ with some partition points s_i added

$$\begin{aligned} &\Big\|\sum_{j=1}^{m} \sum_{i=i_{j-1}+1}^{i_j} \frac{\zeta^2}{2} \cdot \Big[\frac{\partial^2 F}{\partial x^2}(s_{i-1},\, \xi_i^*) - \frac{\partial^2 F}{\partial x^2}(u_{j-1},\, \xi_{u_{j-1}})\Big] \cdot (W_{s_i} - W_{s_{i-1}})^2\Big\|_2 \\ &\qquad \le \frac{C^2}{2}\, \Big(\varepsilon \cdot \big(2 \sup\big|\frac{\partial^2 F}{\partial x^2}\big|\big)^4 + \varepsilon^4\Big)^{1/4} \cdot (105)^{1/4} \cdot \sum_{i=1}^{n}(s_i - s_{i-1}) \\ &\qquad = \mathrm{const} \cdot \Big(\varepsilon \cdot \big(2 \sup\big|\frac{\partial^2 F}{\partial x^2}\big|\big)^4 + \varepsilon^4\Big)^{1/4}. \end{aligned} \tag{34.14}$$

This proves our statement, the last one in the theorem we wanted to prove.

We have proved our Itô's formula under some restrictions; let us formulate the result under minimum restrictions (but we are not going to give the proof; its ideas are outlined at the beginning of this section):

Theorem 34.1. *Let ξ_t, $t \ge t_0$, be a stochastic process with stochastic differential*

$$d\,\xi_t = f(t,\, \omega)\, dt + g(t,\, \omega)\, dW_t, \tag{34.15}$$

with random functions $f(t,\, \omega)$, $g(t,\, \omega)$ being progressively measurable, $f(t,\, \omega)$ Lebesgue integrable, $g(t,\, \omega)$ with finite $\int_{t_0}^{t} E g(s,\, \omega)^2\, ds$; let the function $F(t,\, x)$ be once continuously differentiable in t and twice in x, and suppose that

$$\int_{t_0}^{t} E\Big(\frac{\partial F}{\partial x}(s,\, \xi_s) \cdot g(s,\, \omega)\Big)^2\, ds < \infty. \tag{34.16}$$

Then almost surely

$$\begin{aligned} &F(t,\, \xi_t) - F(t_0,\, \xi_{t_0}) \\ &\quad = \int_{t_0}^{t} \Big[\frac{\partial F}{\partial t}(s,\, \xi_s) + \frac{\partial F}{\partial x}(s,\, \xi_s) \cdot f(s,\, \omega) + \frac{1}{2}\, \frac{\partial^2 F}{\partial x^2}(s,\, \xi_s) \cdot g(s,\, \omega)^2\Big]\, ds \\ &\qquad + \int_{t_0}^{t} \frac{\partial F}{\partial x}(s,\, \xi_s) \cdot g(s,\, \omega)\, dW_s. \end{aligned} \tag{34.17}$$

In Theorems 33.1, 34.1 I supposed that a function $F(t,\, x)$, $t \ge t_0$, has a continuous partial derivative $\dfrac{\partial F}{\partial t}(t,\, x)$, $t \ge t_0$, $x \in \mathbb{R}^1$. But a function defined only for $t \ge t_0$ cannot have a (partial) derivative at $t = t_0$: it can have only a one-sided derivative. So what I meant was that the partial derivative exists for $t > t_0$, at $t = t_0$ a one-sided derivative exists, and the function defined as the derivative for $t > t_0$, and as the right-hand derivative for $t = t_0$ is continuous for $t \ge t_0$, $x \in \mathbb{R}^1$. It can be expressed also thus: for every $x \in \mathbb{R}^1$ there exist finite limits

$$\lim_{t \to t_0^+,\, y \to x} \frac{\partial F}{\partial t}(t,\, y) \tag{34.18}$$

(and we can use the notation $\dfrac{\partial F}{\partial t}(t_0,\, x)$ for this limit). This is enough for what we used about the partial derivative (the formula $F(t',\, y) - F(t,\, y) = \dfrac{\partial F}{\partial t}(t^*,\, y) \cdot (t' - t)$, where t^* is between t and t').

What should the multidimensional Itô's formula look like? Suppose $\boldsymbol{\xi}_t$, $t \ge t_0$, is a d-dimensional stochastic process with stochastic differentials of its coordinates

$$d\xi_t^i = f_i(t,\, \omega)\, dt + \sum_{k=1}^{r} g_{ik}(t,\, \omega)\, dW_t^k, \qquad i = 1, ..., d; \tag{34.19}$$

and $F(t,\, \boldsymbol{x}) = F(t,\, x_1, ..., x_d)$ is a smooth function. What should be the terms with the second derivatives in the multidimensional Itô formula

$$dF(t,\, \boldsymbol{\xi}_t) =$$
$$\frac{\partial F}{\partial t}(t,\, \boldsymbol{\xi}_t)\, dt + \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(t,\, \boldsymbol{\xi}_t) \cdot f_i(t,\, \omega)\, dt + \sum_{i=1}^{d} \sum_{k=1}^{r} \frac{\partial F}{\partial x_i}(t,\, \boldsymbol{\xi}_t) \cdot g_{ik}(t,\, \omega)\, dW_t^k + ...?? \tag{34.20}$$

In the spirit of our preliminary investigation before writing the one-dimensional Itô formula, "exotic" stochastic integral should arise here with integrating with respect to $dW_t^k dW_t^l$. This definitely should have something to do with Problem $\boxed{\mathbf{61}}$ about the limit of $\sum_{i=1}^{n} (W_{t_i}^1 - W_{t_{i-1}}^1)(W_{t_i}^2 - W_{t_{i-1}}^2)$.

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For the time being, I am postponing writing the general Itô formula for multidimensional processes with stochastic differentials; but we can write a compromise formula with the process $\boldsymbol{\xi}_t = (\xi_t^1, ..., \xi_t^d)$ being multidimensional if all its components have stochastic differentials with the same one-dimensional Wiener process W_t:

$$d\xi_t^i = f_i(t,\, \omega)\, dt + g_i(t,\, \omega)\, dW_t. \tag{34.21}$$

Namely, if $F(t,\, x_1, ..., x_d)$ is a function that is continuously differentiable once in time t and twice in the spatial arguments $x_1, ..., x_d$ (and some more conditions are satisfied that ensure the existence of the integrals), then we have:

$$\begin{aligned} &dF(t,\, \boldsymbol{\xi}_t) \\ &\quad= \Big[\frac{\partial F}{\partial t}(t,\, \boldsymbol{\xi}_t) + \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(t,\, \boldsymbol{\xi}_t)\cdot f_i(t,\, \omega) + \frac{1}{2}\sum_{i,\, j=1}^{d} \frac{\partial^2 F}{\partial x_i \partial x_j}(t,\, \boldsymbol{\xi}_t)\cdot g_i(t,\, \omega)\, g_j(t,\, \omega)\Big]\, dt \\ &\qquad\qquad + \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(t,\, \boldsymbol{\xi}_t)\cdot g_i(t,\, \omega)\, dW_t. \end{aligned} \tag{34.22}$$

The "preliminary investigation" and the proof are exactly the same as in the one-dimensional case, except that you have more to write.

If a stochastic process ξ_t is a solution of the stochastic equation

$$d\xi_t = b(\xi_t)\, dt + \sigma(\xi_t)\, dW_t, \tag{34.23}$$

we can write for a smooth function $u(t,\, x)$, using Itô's formula:

$$du(t,\, \xi_t) = \Big[\frac{\partial u}{\partial t}(t,\, \xi_t) + b(\xi_t)\cdot \frac{\partial u}{\partial x}(t,\, \xi_t) + \frac{1}{2} a(\xi_t)\cdot \frac{\partial^2 u}{\partial x^2}(t,\, \xi_t)\Big]\, dt + \frac{\partial u}{\partial x}(t,\, \xi_t)\cdot \sigma(\xi_t)\, dW_t, \tag{34.24}$$

where $a(x) = \sigma(x)^2$; which means the following integral equality for $t \ge t_0$:

$$\begin{aligned} &u(t,\, \xi_t) = u(t_0,\, \xi_{t_0}) + \\ &\int_{t_0}^{t} \Big[\frac{\partial u}{\partial t}(s,\, \xi_s) + b(\xi_s)\cdot \frac{\partial u}{\partial x}(s,\, \xi_s) + \frac{1}{2} a(\xi_s)\cdot \frac{\partial^2 u}{\partial x^2}(s,\, \xi_s)\Big]\; ds + \frac{\partial u}{\partial x}(s,\, \xi_s)\cdot \sigma(\xi_s)\; dW_s. \end{aligned} \tag{34.25}$$

It follows from (34.25) that

$$\eta_t = u(t,\, \xi_t) - \int_{t_0}^{t} \Big[\frac{\partial u}{\partial t}(t,\, \xi_t) + Lu(t,\, \bullet)(\xi_s)\Big]\; ds, \qquad t \ge t_0, \tag{34.26}$$

is a martingale, where L is the linear differential operators that is defined on twice continuously differentiable functions f by

$$Lf(x) = \frac{a(x)}{2}\, f''(x) + b(x) f'(x). \tag{34.27}$$

This, because the stochastic integral is a martingale.

In particular, for a function u of one argument x only,

$$\eta_t = u(\xi_t) - \int_{t_0}^{t} Lu(\xi_s)\; ds \tag{34.28}$$

is a martingale.

Of course, we have encountered this statement under different assumptions: for diffusion processes associated with parabolic partial differential equations; and in the case when u belonged to the domain of the infinitesimal operator of a Markov process, Lu being the value of this operator at the function u. It seems that the time has come to extend the definition of a diffusion process so that it encompasses all these cases.

We'll say that a stochastic process ξ_t is a diffusion process if it is a progressively measurable Markov process (it is supposed that a non-decreasing family of σ-algebras is given in the sample space), almost all its trajectories are continuous, and if the random function (34.26) (or (34.28)) is a martingale for a class of smooth functions u, where L is a differential operator.

We'll be calling the differential operator L the *generator* (or generating differential operator) of the process ξ_t.

We have added to our new class of diffusion processes all solutions of stochastic differential equations (34.23). There arise many problems, in particular, that of existence and uniqueness of solutions of stochastic equations; and whether these solutions form a Markov process. But first let us consider some examples (in the next lecture), for which no existence results are needed: in a rare exception, the solution can be written "explicitly".