Lecture 36. Stochastic integral equations. Existence.

Now we go to stochastic equations. Let $b(x)$, $\sigma(x)$ be two continuous functions. Suppose we want to solve the stochastic differential equation

$$d\xi_t = b(\xi_t)\,dt + \sigma(\xi_t)\,dW_t, \qquad t \ge t_0, \tag{36.1}$$

with the initial condition ξ_{t_0} that is a random variable measurable with respect to $\mathcal{F}_{t_0}$ and such that $E\xi_{t_0}^2 < \infty$.

This is, by definition, the same as the stochastic integral equation

$$\xi_t = \xi_{t_0} + \int_{t_0}^t b(\xi_s)\ ds + \int_{t_0}^t \sigma(\xi_s)\ dW_s \tag{36.2}$$

(and the stochastic process ξ_t is supposed to be progressively measurable and to have, almost surely, continuous trajectories $\xi_t(\omega)$).

Let the functions $b(x)$ and $\sigma(x)$ satisfy a Lipschitz condition

$$|b(x) - b(y)| \le C \cdot |x - y|, \qquad |\sigma(x) - \sigma(y)| \le C \cdot |x - y|. \tag{36.3}$$

It follows from (36.3) that the functions $b(x)$, $\sigma(x)$ grow not faster than linearly:

$$|b(x)|,\ |\sigma(x)| \le K + C \cdot |x|, \tag{36.4}$$

where $K = \max\bigl(|b(0)|,\ |\sigma(0)|\bigr)$.

We are going to prove that if the coefficients satisfy the Lipschitz condition, then a solution ξ_t, $t \ge t_0$, of (36.2) exists.

Note that, in contrast with the case of ordinary differential equations, we cannot solve the equation "backwards", from $t = t_0$ to the left: it turns out to be impossible to define the stochastic integral in the reversed time direction so that it has the same properties.

We are going to do it by successive approximations:

$$\xi_t^{(0)} \equiv \xi_{t_0}, \tag{36.5}$$

$$\xi_t^{(n)} = \xi_{t_0} + \int_{t_0}^t b(\xi_s^{(n-1)})\ ds + \int_{t_0}^t \sigma(\xi_s^{(n-1)})\ dW_s, \quad n = 1, 2, 3, \ldots. \tag{36.6}$$

But, while in the case of ordinary differential (integral) equations everything was simple: every next integral was taken of a continuous function – in the case of a stochastic equation there are many conditions that a random integrand has to satisfy for the stochastic integral, and the Lebesgue integral too, to make sense.

We need to prove that for every $n = 0,\ 1,\ 2,\ \ldots$ the random function $\xi_t^{(n)}$

- is progressively measurable;

• $E(\xi_t^{(n)})^2 < \infty$;

• is mean-square continuous in t (if we want to use the *proved* Theorem 32.1 rather than the fact, mentioned without proof after Theorem 32.2 that every progressively measurable square-integrable random function can be stochastically integrated);

• has almost surely continuous trajectories $\xi_t^{(n)}(\omega)$.

(That is, we need to prove that at each new step we can choose such *a version* of the stochastic integral that these properties are satisfied.)

This is proved by induction.

The basis for the induction is $n = 0$: the random function $\xi_t^{(0)} \equiv \xi_{t_0}$ is clearly progressively measurable, and all remaining properties are satisfied in a trivial way.

Now suppose that the above conditions are satisfied with $n-1$ in lieu of n; we have to check that they are satisfied also with n.

The integrands in both integrals are progressively measurable, so both $\int_{t_0}^{t} b(\xi_s^{(n-1)})\,ds$ and the stochastic integral are progressively measurable: the Lebesgue integral is, for every t, measurable with respect to $\mathcal{F}_t$ by Fubini's Theorem, and it is continuous in t; and the stochastic integral by Theorem 32.4.

About finiteness of $E(\xi_t^{(n)})^2$.

For a random variable η, let us denote $\|\eta\|_2 = \sqrt{E\eta^2}$ (the $\mathbf{L}^2$-norm). What we know about the expectation of the square of a stochastic integral can be written using this norm as

$$\Big\|\int_a^b g(s,\,\omega)\;dW_s\Big\|_2^2 = \int_a^b \|g(s,\,\bullet)\|_2^2\,ds, \qquad \Big\|\int_a^b g(s,\,\omega)\;dW_s\Big\|_2 = \sqrt{\int_a^b \|g(s,\,\bullet)\|_2^2\,ds}\,. \tag{36.7}$$

As for the non-stochastic (Lebesgue) integral, we have

Theorem 36.1. *Let $f(s,\,\omega)$, $a \le s \le b$, be a function that is measurable with respect to $(s,\,\omega)$. Then we have*:

$$\Big\|\int_a^b f(s,\,\bullet)\;ds\Big\|_2 \le \int_a^b \|f(s,\,\bullet)\|_2\,ds. \tag{36.8}$$

The proof of this theorem involves approximating the function h with linear combinations of indicators, and approximating an arbitrary set that is measurable with respect to the product of two σ-algebras with unions of "rectangles" $A \times B$; so better we use the following simpler theorem:

Theorem 36.2. *Under the conditions of the previous theorem, if the random function $f(s,\,\omega)$ is mean-square continuous, and at the same time for almost all ω the function $f(s,\,\omega)$ is continuous in s, then the estimate* (36.8) *holds.*

Proof: There exists the mean-square Riemann integral

$$\int_a^b f(s,\,\bullet)\;ds = \underset{\max(t_i - t_{i-1})\to 0}{\mathrm{l.i.m.}} \sum_{i=1}^{n} f(t_i^*,\,\bullet)\cdot(t_i - t_{i-1}); \tag{36.9}$$

this integral coincides almost surely with the integral $\int_a^b f(s,\,\omega)\;ds$ evaluated separately for every ω (for almost all ω): this is because the mean-square limit coincides almost surely with the almost-sure limit. The $\mathbf{L}^2$-norm of the linear combination $\sum_{i=1}^n f(t_i^*,\,\bullet)\cdot(t_i - t_{i-1})$ does not exceed $\sum_{i=1}^n \|f(t_i^*,\,\bullet)\|_2 \cdot (t_i - t_{i-1})$, which converges as $\max(t_i - t_{i-1}) \to 0$ to the right-hand side of (36.9).

So we have:

$$\|\xi_t^{(n)}\|_2 \le \|\xi_{t_0}\|_2 + \int_{t_0}^t \|b(\xi_s^{(n-1)})\|_2\;ds + \sqrt{\int_{t_0}^t \|\sigma(\xi_s^{(n-1)})\|_2^2\,ds}\,. \tag{36.10}$$

Since $|b(x)|,\ |\sigma(x)| \le K + C\cdot|x|$, we have:

$$\int_{t_0}^t \|b(\xi_s^{(n-1)})\|_2\;ds \le \int_{t_0}^t [K + C\cdot\|\xi_s^{(n-1)}\|_2]\;ds < \infty, \tag{36.11}$$

$$\int_{t_0}^t \|\sigma(\xi_s^{(n-1)})\|_2^2\;ds \le \int_{t_0}^t [K + C\cdot\|\xi_s^{(n-1)}\|_2]^2\;ds < \infty, \tag{36.12}$$

and the $\mathbf{L}^2$-norm (36.10) is finite.

Mean-square continuity: for $t_0 \le t \le t' \le t_0 + T$

$$\begin{aligned}
\|\xi_{t'}^{(n)} - \xi_t^{(n)}\|_2 &= \Big\|\int_t^{t'} b(\xi_s^{(n-1)})\;ds + \int_t^{t'} \sigma(\xi_s^{(n-1)})\;dW_s\Big\|_2\\
&\le \int_t^{t'} \|b(\xi_s^{(n-1)})\|_2\;ds + \sqrt{\int_t^{t'} \|\sigma(\xi_s^{(n-1)})\|_2^2\,ds}\\
&\le \int_t^{t'} [K + C\cdot\|\xi_s^{(n-1)}\|_2]\;ds + \sqrt{\int_t^{t'} [K + C\|\xi_s^{(n-1)}\|_2]^2\,ds}\\
&\to 0 \qquad (t' - t \to 0).
\end{aligned} \tag{36.13}$$

Now we go to the estimates of differences of successive approximations, similar to (35.29). It's enough to have these estimates in every finite time interval $[t_0,\,t_0 + T]$.

We are going to prove that for every $n = 1,\,2,\,3,\,\ldots$ and every $t \in [t_0,\,t_0 + T]$

$$\|\xi_t^{(n)} - \xi_t^{(n-1)}\|_2 \le C_1 \cdot C_2^{n-1} \cdot \frac{(t - t_0)^{n/2}}{\sqrt{n!}}, \tag{36.14}$$

where $C_1,\,C_2$ are some constants; or, which is the same,

$$\|\xi_t^{(n)} - \xi_t^{(n-1)}\|_2^2 \le C_1^2 \cdot C_2^{2(n-1)} \cdot \frac{(t - t_0)^n}{n!}. \tag{36.15}$$

We'll prove (36.14), (36.15) by induction. The basis is $n = 1$:

$$\begin{aligned}
\|\xi_t^{(1)} - \xi_t^{(0)}\|_2 &= \Big\| \int_{t_0}^t b(\xi_{t_0})\ ds + \int_{t_0}^t \sigma(\xi_{t_0})\ dW_s \Big\|_2 \\
&\le \int_{t_0}^t \|b(\xi_{t_0})\|_2\ ds + \sqrt{\int_{t_0}^t \|\sigma(\xi_{t_0})\|_2^2\, ds} \qquad (36.16) \\
&\le (K + \|\xi_{t_0}\|_2) \cdot (t - t_0) + \sqrt{(K + \|\xi_{t_0}\|_2)^2 \cdot (t - t_0)} \\
&\le C_1 \cdot (t - t_0)^{1/2},
\end{aligned}$$

where

$$C_1 = (K + \|\xi_{t_0}\|_2) \cdot (1 + \sqrt{T}); \qquad (36.17)$$

and (36.14), (36.15) is satisfied for $n = 1$.

Suppose these inequalities hold for $n-1$ instead of n; let us estimate $\|\xi_t^{(n)} - \xi_t^{(n-1)}\|_2$. We have:

$$\begin{aligned}
&\|\xi_t^{(n)} - \xi_t^{(n-1)}\|_2 \\
&\quad = \Big\| \int_{t_0}^t [b(\xi_s^{(n-1)}) - b(\xi_s^{(n-2)})]\ ds + \int_{t_0}^t [\sigma(\xi_s^{(n-1)}) - \sigma(\xi_s^{(n-2)})]\ dW_s \Big\|_2 \\
&\quad \le \int_{t_0}^t \|b(\xi_s^{(n-1)}) - b(\xi_s^{(n-2)})\|_2\ ds + \sqrt{\int_{t_0}^t \|\sigma(\xi_s^{(n-1)}) - \sigma(\xi_s^{(n-2)})\|_2^2\, ds} \qquad (36.18) \\
&\quad \le C \cdot \int_{t_0}^t \|\xi_s^{(n-1)} - \xi_s^{(n-2)}\|_2\ ds + \sqrt{C^2 \cdot \int_{t_0}^t \|\xi_s^{(n-1)} - \xi_s^{(n-2)}\|_2^2\, ds}.
\end{aligned}$$

By the Schwarz inequality, the first integral does not exceed

$$\sqrt{\int_{t_0}^t 1\ ds \cdot \int_{t_0}^t \|\xi_s^{(n-1)} - \xi_s^{(n-2)}\|_2^2\, ds} = \sqrt{T} \cdot \sqrt{\int_{t_0}^t \|\xi_s^{(n-1)} - \xi_s^{(n-2)}\|_2^2\, ds}, \qquad (36.19)$$

and

$$\|\xi_t^{(n)} - \xi_t^{(n-1)}\|_2^2 \le (1 + \sqrt{T})^2 \cdot C^2 \int_{t_0}^t \|\xi_s^{(n-1)} - \xi_s^{(n-2)}\|_2^2\, ds. \qquad (36.20)$$

So from (36.16) we get:

$$\|\xi_t^{(2)} - \xi_t^{(1)}\|_2^2 \le (1 + \sqrt{T})^2 \cdot C^2 \cdot \int_{t_0}^t C_1^2 (s - t_0)\ ds = (1 + \sqrt{T})^2 \cdot C^2 \cdot C_1^2 \cdot \frac{(t - t_0)^2}{2}, \qquad (36.21)$$

$$\|\xi_t^{(3)} - \xi_t^{(2)}\|_2^2 \le (1 + \sqrt{T})^4 \cdot C^4 \cdot C_1^2 \cdot \frac{(t - t_0)^3}{3!}, \qquad (36.22)$$

in general,

$$\|\xi_t^{(n)} - \xi_t^{(n-1)}\|_2^2 \le \big((1 + \sqrt{T})^2 \cdot C^2\big)^{n-1} \cdot C_1^2 \cdot \frac{(t - t_0)^n}{n!}; \qquad (36.23)$$

that is, (36.14), (36.15) are satisfied with $C_2 = C(1+\sqrt{T})$.

Now, the series

$$\sum_{n=1}^{\infty} C_1 \cdot C_2^{n-1} \cdot \frac{(t-t_0)^{n/2}}{\sqrt{n!}}, \tag{36.24}$$

whose terms dominate $\|\xi_t^{(n)} - \xi_t^{(n-1)}\|_2$, converges (uniformly in $t \in [t_0,\, t_0+T]$), even though its sum is not expressed in terms of an exponential function, as in (35.30); therefore there exists the mean-square limit

$$\xi_t = \operatorname*{l.i.m.}_{n\to\infty} \xi_t^{(n)}, \tag{36.25}$$

progressively measurable, and having a finite quadratic expectation. This convergence is uniform in $t \in [t_0,\, t_0+T]$:

$$\max_{t_0 \le t \le t_0+T} \|\xi_t^{(n)} - \xi_t\|_2 \le \sum_{i=n+1}^{\infty} C_1 \cdot C_2^{i-1} \cdot \frac{(t-t_0)^{i/2}}{\sqrt{i!}} \to 0 \qquad (n\to\infty); \tag{36.26}$$

it follows from this that the limiting random function is mean-square continuous, and the stochastic integral $\int_{t_0}^{t} \sigma(\xi_s)\, dW_s$ makes sense.

For the limit ξ_t we have two of the four things that we need: square-integrability and mean-square continuity; we need to prove that such a version of ξ_t can be chosen that it is almost surely continuous for all $t \ge t_0$, and that $\xi_t(\omega)$ is progressively measurable. We are doing this using the fact that the right-hand side of the estimate (36.26) decreases (faster than) exponentially. We have dona a similar thing in Theorem 32.4. Let me formulate a general theorem about this:

Theorem 36.3. *Let* $\eta_t^n = \int_{t_0}^{t} g_n(s,\,\omega)\, dW_s$ *for every* n *be a version of the stochastic integral that is almost surely continuous in* t *and progressively measurable*; *suppose* $\int_{t_0}^{t_0+T} \|g_n(s,\,\bullet) - g_{n-1}(s,\,\bullet)\|_2^2\, ds < q^n$, *where* $q<1$.

Then almost surely the sequence η_t^n, $t_0 \le t \le t_0+T$, *converges uniformly as* $n\to\infty$; *and* $\eta_t(\omega) = \lim_{n\to\infty} \eta_t^n(\omega)$ *(replaced by* 0 *if a finite limit does not exist) is an almost surely continuous and progressively measurable version of the stochastic integral* $\int_{t_0}^{t} g(s,\,\omega)\, dW_s$, *where* $g(s,\,\omega)$ *is the* $\mathbf{L}^2$*-limit of* $g_n(s,\,\omega)$.

The **proof**, just as that of Theorem 32.4, is based on the use of Kolmogorov inequality ($1/2^n$ is replaces by $q^{n/3}$).

To handle the first integral in (36.2), which we understand as a Lebesgue integral $\int_{t_0}^{t} b\big(\xi_s(\omega)\big)\, ds$, taken separately for each (rather, *almost* each) $\omega \in \Omega$, it's good to have our trajectories $\xi_t(\omega)$ continuous for all t for almost all ω (almost surely); and for this, almost-sure uniform convergence of the trajectories $\xi_t^{(n)}$ is sufficient.

So as $n \to \infty$, the random functions $\xi_t^{(n-1)}$ almost surely converge to ξ_t uniformly for $t \in [t_0, t_0 + T]$; so the Lebesgue integral $\int_{t_0}^{t} b(\xi_s^{(n-1)})\, ds$ converges almost surely to $\int_{t_0}^{t} b(\xi_s)\, ds$. The stochastic integral $\int_{t_0}^{t} \sigma(\xi_s^{(n-1)})\, dW_s$ converges in the mean squares to $\int_{t_0}^{t} \sigma(\xi_s)\, dW_s$. From both of these kinds of convergence follows convergence in probability; so, as $n \to \infty$, the right-hand side of (36.6) converges in probability to the right-hand side of (36.2). And the left-hand side of (36.6) converges to that of (36.2); so by the almost-uniqueness of the limit in probability we have that the equation (36.2) is satisfied almost surely.

The existence of a solution of a stochastic equation, under the Lipschitz condition on the coefficients $b(x)$, $\sigma(x)$, is proved.