Lecture 37. Stochastic integral equations. Uniqueness. The solution as a functional of the Wiener process.

Now to the uniqueness.

Suppose that ξ_t, $t \ge t_0$, is a mean-square continuous solution of (36.2) with $E\xi_t^2 < \infty$, and η_t is a mean-square continuous solution of the same equation with a different initial condition:

$$\eta_t = \eta_{t_0} + \int_{t_0}^t b(\eta_s)\,ds + \int_{t_0}^t \sigma(\eta_s)\,dW_s, \tag{37.1}$$

$E\eta_{t_0}^2 < \infty$. We are going to prove that for $t_0 \le t \le t_0 + T$

$$\|\xi_t - \eta_t\|_2 \le \|\xi_{t_0} - \eta_{t_0}\|_2 \cdot F(t), \tag{37.2}$$

where $F(t)$ is some continuous function; which means that

$$E(\xi_t - \eta_t)^2 \le E(\xi_{t_0} - \eta_{t_0})^2 \cdot F(t)^2. \tag{37.3}$$

We have:

$$\begin{aligned}\|\xi_t - \eta_t\|_2 &= \|\xi_{t_0} - \eta_{t_0}\|_2 + \left\|\int_{t_0}^t [b(\xi_s) - b(\eta_s)]\,ds + \int_{t_0}^t [\sigma(\xi_s) - \sigma(\eta_s)]\,dW_s\right\|_2 \\ &\le \|\xi_{t_0} - \eta_{t_0}\|_2 + \int_{t_0}^t \|b(\xi_s) - b(\eta_s)\|_2\,ds + \sqrt{\int_{t_0}^t \|\sigma(\xi_s) - \sigma(\eta_s)\|_2^2\,ds} \\ &\le \|\xi_{t_0} - \eta_{t_0}\|_2 + C\cdot\left[\int_{t_0}^t \|\xi_s - \eta_s\|_2\,ds + \sqrt{\int_{t_0}^t \|\xi_s - \eta_s\|_2^2\,ds}\right] \\ &\le \|\xi_{t_0} - \eta_{t_0}\|_2 + C\cdot(1+\sqrt{T})\sqrt{\int_{t_0}^t \|\xi_s - \eta_s\|_2^2\,ds}\,.\end{aligned} \tag{37.4}$$

Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain:

$$E(\xi_t - \eta_t)^2 \le 2\|\xi_{t_0} - \eta_{t_0}\|_2 + 2C^2\cdot(1+\sqrt{T})^2\cdot\int_{t_0}^t E(\xi_s - \eta_s)^2\,ds. \tag{37.5}$$

The function $\|\xi_t - \eta_t\|_2^2$ satisfies the integral inequality of the same kind as (35.36), with $2\|\xi_{t_0} - \eta_{t_0}\|_2^2$ instead of $\|\xi_{t_0} - \eta_{t_0}\|_2$ and $2C^2\cdot(1+\sqrt{T})^2$ instead of C; so, similarly to (35.37), (35.40), we have:

$$\|\xi_t - \eta_t\|_2^2 \le 2\|\xi_{t_0} - \eta_{t_0}\|_2^2 \cdot e^{2C^2(1+\sqrt{T})^2(t-t_0)}. \tag{37.6}$$

This proves the uniqueness of a solution with finite $E\xi_t^2$; and at the same time, mean-square continuous dependence of the solution on the initial condition.

The same results of existence and uniqueness are proved in the same way for *systems* of stochastic equations

$$d\xi_t^i = b_i(\boldsymbol{\xi}_t)\,dt + \sum_{k=1}^{r} \sigma_{ik}(\boldsymbol{\xi}_t)\,dW_t^k, \qquad i = 1, ..., d, \tag{37.7}$$

with initial conditions $\xi_{t_0}^i$ or, in the form of integral equations,

$$\xi^i(t) = \xi_{t_0}^i + \int_{t_0}^{t} b_i(\boldsymbol{\xi}_s)\ ds + \sum_{k=1}^{r} \int_{t_0}^{t} \sigma_{ik}(\boldsymbol{\xi}_s)\ dW_s^k, \qquad i = 1, ..., d; \tag{37.8}$$

or, in the vector-matrix form,

$$d\boldsymbol{\xi}_t = \boldsymbol{b}(\boldsymbol{\xi}_t)\,dt + \sigma(\boldsymbol{\xi}_t)\,d\boldsymbol{W}_t \tag{37.9}$$

with the initial condition $\boldsymbol{\xi}_{t_0}$,

$$\boldsymbol{\xi}_t = \boldsymbol{\xi}_{t_0} + \int_{t_0}^{t} \boldsymbol{b}(\boldsymbol{\xi}_s)\ ds + \int_{t_0}^{t} \sigma(\boldsymbol{\xi}_s)\ d\boldsymbol{W}_s, \tag{37.10}$$

if the coefficients, the column vector-valued function $\boldsymbol{b}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, with components $b_1(\boldsymbol{x})$, ..., $b_d(\boldsymbol{x})$ and the $(d \times r)$-matrix valued $\sigma(\boldsymbol{x}) = \big(\sigma_{ik}(\boldsymbol{x})\big)_{\substack{i=1,\,...,\,d\\ k=1,\,...,\,r}}$ satisfy Lipschitz conditions in $\boldsymbol{x}$:

$$|b_i(\boldsymbol{x}) - b_i(\boldsymbol{y})| \le C \cdot |\boldsymbol{x} - \boldsymbol{y}|, \qquad |\sigma_{ik}(\boldsymbol{x}) - \sigma_{ik}(\boldsymbol{y})| \le C \cdot |\boldsymbol{x} - \boldsymbol{y}| \tag{37.11}$$

The proof is exactly the same; but, of course, writing all the formulas would take too much space in this case.

Now I want to represent the results obtained in some other forms (I present it here in a different order, and with notations different from those used in the lecture), using a higher level of abstraction. I did not formulate the results about existence and uniqueness as separate theorems; to compensate for it, I'll formulate several smaller theorems in what follows. For simplicity of notations I am going to consider the one-dimensional case. Also for simplicity let us start with the solution ξ_t, $t \ge 0$, of our stochastic equation with an initial condition ξ_0 at time 0 (ξ_0 is supposed to be $\mathcal{F}_0$-measurable).

I want to prove that this solution can be represented as some (measurable) function of two random variables: the initial condition $\xi_0(\omega)$, taking values in the measurable space $(\mathbb{R}^1, \mathcal{B}^1)$, and the trajectory $W_\bullet(\omega)$ of the Wiener process, taking values in the space $\mathbf{C} = \mathbf{C}\big([0, \infty), \mathbb{R}^1\big)$ of continuous functions u_t, $t \ge 0$ (I am asking forgiveness for denting with $\mathbf{C}$ sometimes the space of all continuous functions, and sometimes that of *bounded* continuous functions; here it is definitely *all* continuous functions, bounded and unbounded. As we know, the trajectories of the Wiener process on the time interval from 0 to ∞ are almost surely *un*bounded).

But here you should have started shouting: "It's nonsense! We can speak of random variables taking values in a *measurable space* $(X, \mathcal{X})$: the space *plus* a σ-algebra on it;

and here you have only the space $\mathbf{C}$, and no σ-algebra!" This is true: before speaking of a random variable $W_\bullet$, I must introduce a σ-algebra in my space $\mathbf{C}$.

As this σ-algebra – let me denote it $\mathcal{C}$ – I take the σ-algebra generated by all subsets of $\mathbf{C}$ described by the values of a function at finitely many points:

$$\mathcal{C} = \sigma\big\{\{u_\bullet \in \mathbf{C}\colon (u_{t_1}, ..., u_{t_n}) \in C\}, t_1, ..., t_n \ge 0, C \in \mathcal{B}^n, n \ge 1\big\}. \tag{37.12}$$

Let us check that the trajectory $W_\bullet$ of the Wiener process is a random variable with values in $(\mathbf{C}, \mathcal{C})$.

This means that for every set $A \in \mathcal{C}$ the set

$$\{\omega\colon W_\bullet(\omega) \in A\} \tag{37.13}$$

belongs to the σ-algebra $\mathcal{F}$ of all events.

Let us denote

$$\mathcal{E} = \big\{\{u_\bullet \in \mathbf{C}\colon (u_{t_1}, ..., u_{t_n}) \in C\}, t_1, ..., t_n \ge 0, C \in \mathcal{B}^n, n \ge 1\big\}; \tag{37.14}$$

and $\mathcal{D}$ will be the class of all sets A for which the ω-set (37.13) belongs to $\mathcal{F}$.

The class $\mathcal{E} \subseteq \mathcal{D}$. Indeed, this means that for every n, t_1, ..., $t_n \ge 0$, and $C \in \mathcal{B}^n$, $A = \{u_\bullet \in \mathbf{C}\colon (u_{t_1}, ..., u_{t_n}) \in C\}$ we have:

$$\{\omega\colon W_\bullet(\omega) \in A\} = \big\{\omega\colon \big(W_{t_1}(\omega), ..., W_{t_n}(\omega)\big) \in C\big\} \in \mathcal{F}. \tag{37.15}$$

But this is obvious, because W_{t_1}, ..., W_{t_n} are random variables. So: $\mathcal{D} \supseteq \mathcal{E}$.

The next thing: the class $\mathcal{D}$ is a σ-algebra in $\mathbf{C}$. Indeed, it contains $\mathbf{C}$, because the set $\{\omega\colon W_\bullet(\omega) \in \mathbf{C}\} = \Omega$; if A belongs to $\mathcal{D}$, so does the complement A^c: $\{\omega\colon W_\bullet(\omega) \in A^c\} = \{\omega\colon W_\bullet(\omega) \in A\}^c \in \mathcal{F}$; and closedness of $\mathcal{D}$ under countable unions is equally easy.

So $\mathcal{D}$ contains the smallest σ-algebra $\mathcal{C}$ containing $\mathcal{E}$; or, in the opposite direction: $\mathcal{C} \subseteq \mathcal{D}$, which proves our statement.

Of course, we can do it not only with a Wiener process, but with every stochastic process with continuous trajectories; and if the trajectories are not continuous, but all of them belong to some function space $\mathbf{S}$, we can consider the trajectory of our stochastic process as a random point taking values in the measurable space $(\mathbf{S}, \mathcal{S})$, where the σ-algebra $\mathcal{S}$ is defined by formula (37.12) in which $\mathbf{C}$ is replaced with $\mathbf{S}$. For the trajectories we can consider their *distributions* in function spaces, and so on. Considering distributions in functional spaces is very interesting; but not now: we are interested in stochastic equations right now.

Theorem 37.1. *If the coefficients b, σ satisfy a Lipschitz condition, there exists a function* (functional?) $\Xi_t^x(u_\bullet)$ (Ξ is the capital Greek letter "xi") *of* $x \in \mathbb{R}^1$, $t \in [0, \infty)$, *and* $u_\bullet \in \mathbf{C}$, *measurable with respect to* $\mathcal{B}^1 \times \mathcal{B}_{[0,\infty)} \times \mathcal{C}$, *such that a version of the solution of the equation* $d\xi_t = b(\xi_t)\,dt + \sigma(\xi_t)\,dW_t$ *with the initial condition* ξ_0 *at time* 0 *is given by*

$$\xi_t(\omega) = \Xi_t^{\xi_0}\big(W_\bullet(\omega)\big). \tag{37.16}$$

In the **proof**, we have to repeat once more the whole theory of stochastic integrals and the whole proof of existence for stochastic equations, but in a new situation, on a more abstract level.

Let $\mathfrak{T}$ be the partition $t_0 = 0 < t_1 < t_2 < ... < t_i < ...$ of the interval $[0, \infty)$. For a step function of the form

$$g^x(t, u_\bullet) = \sum_i c_i^x(u_\bullet) \cdot I_{[t_{i-1}, t_i)}(t) \tag{37.17}$$

we define its stochastic integral, for $t_{j-1} \le t \le t_j$, by

$$\int_0^t g^x(s, u_\bullet)\, du_s = \sum_{i=1}^{j-1} c_i^x(u_\bullet) \cdot (u_{t_i} - u_{t_{i-1}}) + c_j^x(u_\bullet) \cdot (u_t - u_{t_{j-1}}). \tag{37.18}$$

This integral is clearly measurable in the variables $(x, t, u_\bullet)$.

Obviously, if we take here $u_s = W_s(\omega)$ and $x = \xi_0$, we obtain our standard definition of the stochastic integral of a step random function $\int_0^t g^{\xi_0}\big(s, W_\bullet(\omega)\big)\, dW_s$.

Now let $g^x(t, u_\bullet)$ be a non-step function of $x \in \mathbb{R}^1$, $t \in [0, \infty)$, and $u_\bullet \in \mathbf{C}$, measurable with respect to $\mathcal{B}^1 \times \mathcal{B}_{[0, \infty)} \times \mathcal{C}$. Let us consider the partition $\mathfrak{T}_n$ with partition points $t_0^n = 0 < t_1^n < t_2^n < ... < t_i^n < ...$ with the lengths of the intervals equal to $1/10^n$. We define step functions $g_n^x(t, u_\bullet)$ by

$$g_n^x(t, u_\bullet) = g^x(t_{i-1}^n, u_\bullet) \qquad \text{for } t_{i-1}^n \le t < t_i^n; \tag{37.19}$$

and we define the stochastic integral of the function g^x by

$$\int_0^t g^x(s, u_\bullet)\, du_s = \begin{cases} \lim\limits_{n\to\infty} \int_0^t g_n^x(s, u_\bullet)\, du_s & \text{if a finite limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{37.20}$$

Again, this limit is measurable in the triple $(x, t, u_\bullet)$.

The stochastic integral $\int_0^t g^{\xi_0}\big(s, W_\bullet(\omega)\big)\, dW_s$ is almost surely equal to this if the random function $g^{\xi_0}\big(s, W_\bullet(\omega)\big)$ is progressively measurable and mean-square continuous with $E\big(g^{\xi_0}\big(t', W_\bullet(\omega)\big) - g^{\xi_0}\big(t, W_\bullet(\omega)\big)\big)^2 < \text{const} \cdot |t' - t|^\beta$ for some $\beta > 0$ (because the limit exists almost surely, see the proof of Theorem 32.4, and the almost-sure limit almost surely coincides with the mean-square limit).

Now let us go to stochastic equations.

For the stochastic integral equation

$$\Xi_t^x(u_\bullet) = x + \int_0^t b\big(\Xi_s^x(u_\bullet)\big)\, ds + \int_0^t \sigma\big(\Xi_s^x(u_\bullet)\big)\, du_s, \tag{37.22}$$

let us write successive approximations: $\Xi_t^{x,(0)}(u_\bullet) \equiv x$,

$$\Xi_t^{x,(n)}(u_\bullet) = x + \int_0^t b\big(\Xi_s^{x,(n-1)}(u_\bullet)\big)\, ds + \int_0^t \sigma\big(\Xi_s^{x,(n-1)}(u_\bullet)\big)\, du_s, \tag{37.23}$$

where the first integral is understood as a Lebesgue one, and is replaced with 0 if the integrand is not integrable. And then we take

$$\Xi_t^x(u_\bullet) = \begin{cases} \lim\limits_{n\to\infty} \Xi_t^{x,\,(n)}(u_\bullet) & \text{if a finite limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{37.24}$$

Again this limit is measurable. And if we take ξ_0 in the place of x, and $W_t(\omega)$ instead of u_t, all limits exist almost surely (see Lecture 36), and this turns out to be a version of the solution $\xi_t(\omega)$ of our stochastic equation.

The following theorem is *very* easy:

Theorem 37.2. *For any constant a we have $\Xi_t^x(u_\bullet - a) = \Xi_t^x(u_\bullet)$; and the solution ξ_t of our equation can be written as $\xi_t = \Xi_t^{\xi_0}(W_\bullet - W_0)$.*

This is because only increments of the Wiener process (of the function $u_\bullet$) are used in the stochastic integrals. The last statement means that we can assume that our Wiener process starts from 0.

Theorem 37.3. *A version of the solution ξ_t, $t \ge t_0$, of our stochastic equation with initial condition ξ_{t_0} at the time t_0 can be written as*

$$\xi_t = \Xi_{t-t_0}^{\xi_{t_0}}(\tilde{W}_\bullet), \tag{37.25}$$

where $\tilde{W}_t = W_{t_0+t} - W_{t_0}$, $t \ge 0$ (this $\tilde{W}_t$ is another Wiener process).

Proof. The successive approximations, and all stochastic integrals (and all Lebesgue ones) that arise in solving our equation with the initial condition at time t_0 involve increments of the Wiener process W_t after time t_0, which are the same as increments of the other Wiener process $\tilde{W}_t$ after time 0: the successive approximations are the same. And $\Xi_{t_0-t_0}^{\xi_{t_0}}(\tilde{W}_\bullet)$ is precisely ξ_{t_0}.

The plan of what is to follow: The random variable ξ_{t_0} is measurable with respect to $\mathcal{F}_{t_0}$, and $\tilde{W}_\bullet$ is independent from this σ-algebra. Using this fact and Problem

65 Let $F(x, y)$ be a bounded $(\mathcal{X} \times \mathcal{Y})$-measurable function on the space $(X \times Y, \mathcal{X} \times \mathcal{Y})$. Let ξ be a random variable with values in the space $(X, \mathcal{X})$, measurable with respect to a σ-algebra $\mathcal{A}$; let η be a random variable with values in $(Y, \mathcal{Y})$ that is independent with the σ-algebra $\mathcal{A}$. Let $G(x) = EF(x, \eta)$.

Prove that $E\big(F(\xi, \eta) \| \mathcal{A}\big) = G(\xi)$ (almost surely, of course),

we prove that the solution ξ_t of the stochastic equation is a Markov process (and so it is a diffusion process, and a Feller one at that – because the solution depends continuously on the initial point).