Lecture 39. Equations $Lu(\boldsymbol{x}) + c(\boldsymbol{x}) \cdot u(\boldsymbol{x}) = g(\boldsymbol{x})$.

When it was about the equation $Lu(\boldsymbol{x}) = g(\boldsymbol{x})$, we considered a martingale $\eta_t = u(\boldsymbol{\xi}_t) - \int_0^t Lu(\xi_s)\, ds$. To handle the equation $Lu + cu = g$, we should try to construct a martingale being a difference of something and an integral involving $Lu(\xi_s) - c(\xi_s) \cdot u(\xi_s)$. Let us start with the case of $c(\boldsymbol{x}) \equiv c = \text{const}$.

Let us find the stochastic differential $d\big(e^{ct} u(\boldsymbol{\xi}_t)\big)$. We have: $e^{ct} u(\boldsymbol{\xi}_t) = F(t, \boldsymbol{\xi}_t)$, where $F(t, \boldsymbol{x}) = e^{ct} u(\boldsymbol{x})$;

$$\begin{aligned} &d\big(e^{ct} u(\boldsymbol{\xi}_t)\big) \\ &\quad = \Big[\frac{\partial F}{\partial t}(t, \boldsymbol{\xi}_t) + \sum_i b_i(\boldsymbol{\xi}_t) \cdot \frac{\partial F}{\partial x_i}(t, \boldsymbol{\xi}_t) + \frac{1}{2} \sum_{i,j} a_{ij}(\boldsymbol{\xi}_t) \cdot \frac{\partial^2 F}{\partial x_i \partial x_j}(\boldsymbol{\xi}_t)\Big]\, dt + \ \dots \ d\boldsymbol{W}_t \\ &\quad = e^{ct} \cdot [c \cdot u(\boldsymbol{\xi}_t) + Lu(\boldsymbol{\xi}_t)]\, dt + \ \dots \ d\boldsymbol{W}_t. \end{aligned} \tag{39.1}$$

So we have that

$$\eta_t = e^{ct} u(\boldsymbol{\xi}_t) - \int_0^t e^{cs} [Lu(\boldsymbol{\xi}_s) + c \cdot u(\boldsymbol{\xi}_s)]\, ds \tag{39.2}$$

is a martingale (if $\int_0^t E\big(e^{cs} \cdot \frac{\partial u}{\partial x_i}(\boldsymbol{\xi}_s) \cdot \sigma_{ik}(\boldsymbol{\xi}_s)\big)^2 ds < \infty$, and a couple of random functions are almost surely Lebesgue integrable).

What if $c(\boldsymbol{x}) \neq \text{const}$? It turns out that we should replace e^{ct} with $\exp\Big\{\int_0^t c(\boldsymbol{\xi}_s)\, ds\Big\}$. Let us find the stochastic differential $d\Big(\exp\Big\{\int_0^t c(\boldsymbol{\xi}_s)\, ds\Big\} \cdot u(\boldsymbol{\xi}_t)\Big)$. To do this, let us introduce the random function

$$\xi_t^0 = \int_0^t c(\boldsymbol{\xi}_s)\, ds, \tag{39.3}$$

and find the stochastic differential of $F(\xi_t^0, \xi_t^1, \dots, \xi_t^d)$, where $F(x_0, x_1, \dots, x_d) = e^{x_0} \cdot u(x_1, \dots, x_d)$. We have:

$$d\xi_t^0 = c(\boldsymbol{\xi}_t)\, dt; \tag{39.4}$$

and we already know the stochastic differentials of other coordinates ξ_t^i. By Itô's formula we have:

$$d\Big(\exp\Big\{\int_0^t c(\boldsymbol{\xi}_s)\, ds\Big\} \cdot u(\boldsymbol{\xi}_t)\Big) = \exp\Big\{\int_0^t c(\boldsymbol{\xi}_s)\, ds\Big\} \cdot [Lu(\boldsymbol{\xi}_t) + c(\boldsymbol{\xi}_t) \cdot u(\boldsymbol{\xi}_t)]\, dt + \ \dots \ d\boldsymbol{W}_t. \tag{39.5}$$

So the random function

$$\eta_t = \exp\Big\{\int_0^t c(\boldsymbol{\xi}_s)\, ds\Big\} \cdot u(\boldsymbol{\xi}_t) - \int_0^t \exp\Big\{\int_0^s c(\boldsymbol{\xi}_v)\, dv\Big\} \cdot [Lu(\boldsymbol{\xi}_s) + c(\boldsymbol{\xi}_s) \cdot u(\boldsymbol{\xi}_s)]\, ds \tag{39.6}$$

is a martingale (under some restrictions).

We can at once apply this to parabolic equations: The value $u(0,\, \boldsymbol{x})$ of the bounded solution of the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,\, \boldsymbol{x}) + Lu(t,\, \bullet)(\boldsymbol{x}) + c(\boldsymbol{x}) \cdot u(t,\, \boldsymbol{x}) = g(t,\, \boldsymbol{x}), \qquad t \le T, \\ &u(T,\, \boldsymbol{x}) = f(\boldsymbol{x}) \end{aligned} \tag{39.7}$$

can be represented as

$$u(0,\, \boldsymbol{x}) = E_{\boldsymbol{x}} \Big[\exp\Big\{ \int_0^T c(\boldsymbol{\xi}_t)\, dt \Big\} \cdot f(\boldsymbol{\xi}_T) - \int_0^T \exp\Big\{ \int_0^t c(\boldsymbol{\xi}_s)\, ds \Big\} \cdot g(t,\, \boldsymbol{\xi}_t)\, dt \Big]. \tag{39.8}$$

The Cauchy problem (39.7) is solved with a *final* condition $u(T,\, \boldsymbol{x}) = f(\boldsymbol{x})$; we are more accustomed to considering *initial* conditions. A simple change of variables allows us to write a formula expressing the solution of a problem solved forward in time from an initial condition.

But better we stick to *elliptic* equations.

When we considered elliptic equations $Lu = g$ in a region G, problems arose with whether the time τ_G of our process leaving the region was finite, and whether $E_{\boldsymbol{x}} \tau_G < \infty$. In our new situation it should be the problems whether

$$E_{\boldsymbol{x}} \int_0^{\tau_G} \exp\Big\{ \int_0^t c(\boldsymbol{\xi}_s)\, ds \Big\}\, dt < \infty, \tag{39.9}$$

$$E_{\boldsymbol{x}} \exp\Big\{ \int_0^{\tau_G} c(\boldsymbol{\xi}_t)\, dt \Big\} < \infty \tag{39.10}$$

(as for τ_G being almost surely finite, we know already how to solve such problems).

If $c(\boldsymbol{x}) \le 0$, the expectation (39.9) is finite if $E_{\boldsymbol{x}} \tau_G < \infty$; and we can formulate the following result:

Theorem 39.1. *Let* $c(\boldsymbol{x}) \le 0$, *and let* $E_{\boldsymbol{x}} \tau_G < \infty$. *Let* $u(\boldsymbol{x})$ *be a solution of the Dirichlet problem*

$$\begin{aligned} Lu(\boldsymbol{x}) + c(\boldsymbol{x}) \cdot u(\boldsymbol{x}) &= g(\boldsymbol{x}), \quad \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) &= \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \partial G. \end{aligned} \tag{39.11}$$

If the solution $u(\boldsymbol{x})$ *can be extended to the whole space to form a function to which Itô's formula can be applied, then*

$$u(\boldsymbol{x}) = E_{\boldsymbol{x}} \Big[\exp\Big\{ \int_0^{\tau_G} c(\boldsymbol{\xi}_t)\, dt \Big\} \cdot \varphi(\boldsymbol{\xi}_{\tau_G}) - \int_0^{\tau_G} \exp\Big\{ \int_0^t c(\boldsymbol{\xi}_s)\, ds \Big\} \cdot g(\boldsymbol{\xi}_t)\, dt \Big]. \tag{39.12}$$

In particular, the formula (39.12) without the integral subtracted expresses the solution of the Dirichlet problem for the equation $Lu + cu = 0$. Solve

Problem 67 For the one-dimensional Wiener process W_t, find the expectation $E_x e^{c\tau_{(a,\, b)}}$, where c is a negative constant and $\tau_{(a,\, b)}$ the first time at which the Wiener process leaves the interval $(a,\, b)$.

If $c(\boldsymbol{x})$ can take positive values, the situation is not as simple as that. Let me formulate a particular result in this case.

Theorem 39.2. *Let $u(\boldsymbol{x})$ be a bounded nonnegative solution of the Dirichlet problem*

$$\begin{aligned} Lu(\boldsymbol{x}) + c(\boldsymbol{x}) \cdot u(\boldsymbol{x}) = 0, \quad & \boldsymbol{x} \in G, \\ u(\boldsymbol{x}) = 1, \qquad & \boldsymbol{x} \in \partial G. \end{aligned} \tag{39.13}$$

If the solution $u(\boldsymbol{x})$ can be extended to the whole space to form a function to which Itô's formula can be applied, then

$$E_{\boldsymbol{x}} \exp\Big\{ \int_0^{\tau_G} c(\boldsymbol{\xi}_t)\, dt \Big\} = u(\boldsymbol{x}). \tag{39.14}$$

Proof. The random function (39.6) is a martingale, and almost all of its trajectories are continuous. So if we take it at the bounded stopping time $\min(\tau_G, t_*)$, we have $E_{\boldsymbol{x}} \eta_{\min(\tau_G, t_*)} = E_{\boldsymbol{x}} \eta_0$, or, using the expression (39.6) for η_t and the equation (39.13):

$$E_{\boldsymbol{x}} \Big[\exp\Big\{ \int_0^{\min(\tau_G, t_*)} c(\boldsymbol{\xi}_t)\, dt \Big\} \cdot u(\boldsymbol{\xi}_{\min(\tau_G, t_*)}) \Big] = u(\boldsymbol{x}). \tag{39.15}$$

Now we take $t_* \to \infty$. We cannot use here the monotone-convergence theorem or the dominated-convergence theorem; but we have Fatou's Lemma (because u is *nonnegative*!):

$$\begin{aligned} E_{\boldsymbol{x}} \lim_{t_* \to \infty} \Big[\exp\Big\{ \int_0^{\min(\tau_G, t_*)} c(\boldsymbol{\xi}_t)\, dt \Big\} \cdot u(\boldsymbol{\xi}_{\min(\tau_G, t_*)}) \Big] &= E_{\boldsymbol{x}} \Big[\exp\Big\{ \int_0^{\tau_G} c(\boldsymbol{\xi}_t)\, dt \Big\} \Big] \\ \le \varliminf_{t_* \to \infty} E_{\boldsymbol{x}} \lim_{t_* \to \infty} \Big[\exp\Big\{ \int_0^{\min(\tau_G, t_*)} c(\boldsymbol{\xi}_t)\, dt \Big\} \cdot u(\boldsymbol{\xi}_{\min(\tau_G, t_*)}) \Big] &= u(\boldsymbol{x}) < \infty. \end{aligned} \tag{39.16}$$

After we have found out that the expectation is finite, we *can* use the dominated-convergence theorem, obtaining the *equality* (39.14).

Problem **68** Let $(a, b) = (-1, 1)$, $\tau = \tau_{(-1, 1)}$ the first time at which the Wiener process leaves this interval. Find the expectation $E_{0.1} e^{15\tau}$ with accuracy up to 0.01 (if the expectation is infinite, explain it).

The stochastic-equations approach is very fruitful applied to some asymptotic problems for diffusion processes and partial differential equations. Let me show you one example.

Let $b(x)$ be a Lipschitz-continuous periodic function with period 1; and let ξ_t^ε, $\varepsilon > 0$, be a family of diffusion processes arising as solutions of the stochastic equations

$$d\xi_t^\varepsilon = b(\xi_t^\varepsilon)\, dt + dW_t. \tag{39.17}$$

The generating operator of the process ξ_t^ε is

$$L^\varepsilon f(x) = \tfrac{1}{2} f''(x) + b(x/\varepsilon)\, f'(x). \tag{39.18}$$

As $\varepsilon \to 0^+$, the drift coefficient oscillates faster and faster, its amplitude remaining the same. Differential operators with rapidly and periodically oscillating coefficients are used in physics to describe physical processes in periodic structures (such as crystals). It's believed that, on a macro scale, such processes are described by differential operators with *constant* coefficients: in our case that for small ε the process ξ_t^ε can be approximated, in some sense, with the solution $\bar{\xi}_t$ of the equation with a constant drift coefficient $\bar{b}$:

$$d\bar{\xi}_t = \bar{b}\,dt + dW_t; \tag{39.19}$$

and the solutions u^ε of partial differential equations with the operator L^ε approach the solutions of the equations with the operator

$$\bar{L}f(x) = \tfrac{1}{2}f''(x) + \bar{b}\,f'(x). \tag{39.20}$$

This "physical" (mathematical, in fact) phenomenon is called *homogenization*: the space-inhomogeneous stochastic process and solutions of space-inhomogeneous differential equations are replaced with space-homogeneous ones (ones with constant coefficients). We can approach such problems using the methods of partial differential equations; or using stochastic equations. The stochastic-equations approach is easier and more "tangible".

It turns out that as the homogenized drift coefficient $\bar{b}$ is equal to the average of the drift over one period:

$$\bar{b} = \int_0^1 b(x)\ dx. \tag{39.21}$$

Let us prove that

$$\underset{\varepsilon\to 0^+}{\text{l.i.m.}}\ \xi_t^\varepsilon = \bar{\xi}_t, \tag{39.22}$$

where $\bar{\xi}_t$ is the solution of equation (39.19) with the same initial condition as ξ_t^ε.

The stochastic differential equations (39.17), (39.19) are short expressions for the integral equations:

$$\xi_t^\varepsilon = \xi_0^\varepsilon + \int_0^t b(\xi_s^\varepsilon/\varepsilon)\ ds + \int_0^t dW_s, \qquad \bar{\xi}_t = \bar{\xi}_0 + \int_0^t \bar{b}\ ds + \int_0^t dW_s. \tag{39.23}$$

Subtracting these equations from one another, we get:

$$\xi_t^\varepsilon - \bar{\xi}_t = \int_0^t [b(\xi_s^\varepsilon/\varepsilon) - \bar{b}]\ ds \tag{39.24}$$

(remember: "with the same initial condition": $\xi_0^\varepsilon = \bar{\xi}_0$).

Now let us define the function $f(x)$ by

$$f(x) = \int [b(x) - \bar{b}]\ dx. \tag{39.25}$$

Since the integral of $b(x) - \bar{b}$ over one period is equal to 0, the function $f(x)$ is periodic with the same period.

The indefinite integral (39.25) is defined not uniquely but up to a constant summand; let us choose this constant so that

$$\int_0^1 f(x)\ dx = 0; \tag{39.26}$$

and define another periodic function $F(x)$ by

$$F(x) = \int f(x)\ dx. \tag{39.27}$$

The function $F(x)$ is twice continuously differentiable and bounded with its first and second derivatives.

Let us apply Itô's formula to the random function $F(\xi_t^\varepsilon/\varepsilon)$:

$$F(\xi_t^\varepsilon/\varepsilon) = F(\xi_0^\varepsilon/\varepsilon) + \int_0^t \big[\frac{1}{\varepsilon}\, F'(\xi_s^\varepsilon/\varepsilon)\cdot b(\xi_s^\varepsilon/\varepsilon) + \frac{1}{2\varepsilon^2}\, F''(\xi_s^\varepsilon/\varepsilon)\big]\ ds + \int_0^t \frac{1}{\varepsilon}\, F'(\xi_s^\varepsilon/\varepsilon)\ dW_s, \tag{39.28}$$

$$\begin{aligned} &\int_0^t F''(\xi_s^\varepsilon/\varepsilon)\ ds = \int_0^t [b(\xi_s^\varepsilon/\varepsilon) - \bar b]\ ds = \\ &\quad 2\varepsilon^2[F(\xi_t^\varepsilon/\varepsilon) - F(\xi_0^\varepsilon/\varepsilon)] - 2\varepsilon\int_0^t F'(\xi_s^\varepsilon/\varepsilon)\cdot b(\xi_s^\varepsilon/\varepsilon)\ ds - 2\varepsilon\int_0^t F'(\xi_s^\varepsilon/\varepsilon)\ dW_s. \end{aligned} \tag{39.29}$$

The first summand is not greater in absolute value than $4\varepsilon^2\cdot\max|F(x)|$, the second than $2\varepsilon\cdot\max|F'(x)|\cdot\max|b(x)|\cdot t$; and the $\mathbf{L}^2$-norm of the third summand is

$$2\varepsilon\sqrt{\int_0^t \|F'(\xi_s^\varepsilon/\varepsilon)\|_2^2\ ds} \le 2\varepsilon\cdot\max|F'(x)|\cdot\sqrt{t}. \tag{39.30}$$

So we have:

$$\begin{aligned} &\|\xi_t^\varepsilon - \bar\xi_t\|_2 = \Big\|\int_0^t b(\xi_s^\varepsilon/\varepsilon)\ ds - \int_0^t \bar b\ ds\Big\|_2 \\ &\quad \le 4\varepsilon^2\cdot\max|F(x)| + 2\varepsilon\cdot\max|F'(x)|\cdot\max|b(x)|\cdot t + 2\varepsilon\cdot\max|F'(x)|\cdot\sqrt{t}\to 0 \end{aligned} \tag{39.31}$$

as $\varepsilon\to 0^+$. So (39.22) is proved.

Formulation in the language of parabolic differential equations: *If* $u^\varepsilon(t,\,x)$ *is the bounded solution of the Cauchy problem*

$$\begin{aligned} &\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u^\varepsilon}{\partial x^2} + b(x/\varepsilon)\cdot\frac{\partial u^\varepsilon}{\partial x}, \quad t>0, \\ &u^\varepsilon(0,\,x) = f(x), \end{aligned} \tag{39.32}$$

then $\lim_{\varepsilon\to 0^+} u^\varepsilon(t,\,x) = \bar u(t,\,x)$, *where* $\bar u(t,\,x)$ *is the bounded solution of the Cauchy problem*

$$\begin{aligned} &\frac{\partial \bar u}{\partial t} = \frac{1}{2}\,\frac{\partial^2 \bar u}{\partial x^2} + \bar b\cdot\frac{\partial \bar u}{\partial x}, \quad t>0, \\ &\bar u(0,\,x) = f(x), \end{aligned} \tag{39.33}$$

that is,

$$\bar u(t,\,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-(y-x-\bar b t)^2/2t}\cdot f(y)\ dy. \tag{39.34}$$

Think about reformulation in the language of *elliptic* equations, which are, in the one-dimensional case, just ordinary differential equations.