Lecture 5. Discrete Markov chains. Limiting probabilities and mean return times.

Theorem 5.1. *If there exists a limit* $\lim_{n\to\infty} p_{yy}^{(n)} = p_y$, *then this limit is equal to* $\dfrac{1}{E_y\tau_y}$ (taking $\dfrac{1}{\infty} = 0$).

Note that we do not suppose that our Markov chain is ergodic, or that all limits $\lim_{n\to\infty} p_{xy}^{(n)}$ are the same, or that these limits even exist: our statement is about just *one state* y.

I was unsure whether I should give the proof since we'll have another theorem, in which the existence of the limit is not presupposed, but proved; also the proof of that theorem will provide us with the value $1/E_y\tau_y$ for it. But still I'll give the proof, hoping it will be much simpler than the proof of the bigger theorem I only mentioned here.

Proof. First of all, the statement of the theorem is true if the state y is transient: $E_y\tau_y = \infty$ since the random variable takes the value ∞ with positive probability; and $\lim_{n\to\infty} p_{yy}^{(n)} = 0$, because the series $\sum_{n=0}^{\infty} p_{yy}^{(n)}$ converges. What remains is the case of the state y being recurrent: $P_y\{\tau_y < \infty\} = 1$.

Let us prove that

$$P_{yy}(s) = \frac{p_y}{1-s} + o(1-s) \qquad (s \to 1^-). \tag{5.1}$$

This means that for every positive ε there exists an $s_0 < 1$ such that for every $s \in [s_0,\, 1)$ we have

$$\left|P_{yy}(s) - \frac{p_y}{1-s}\right| < \frac{\varepsilon}{1-s}. \tag{5.2}$$

Let us choose a natural n_0 so that $|p_{yy}^{(n)} - p_y| < \varepsilon/2$. The difference in the left-hand side of (5.1) can be rewritten as

$$\sum_{n=0}^{\infty} p_{yy}^{(n)} \cdot s^n - \sum_{n=0}^{\infty} p_y \cdot s^n = \sum_{n=0}^{n_0-1} (p_{yy}^{(n)} - p_y) \cdot s^n + \sum_{n=n_0}^{\infty} (p_{yy}^{(n)} - p_y) \cdot s^n. \tag{5.3}$$

Each summand in the first sum is not greater than 1 in absolute value, so this sum does not exceed n_0. the second sum is not greater in absolute value than $\dfrac{\varepsilon/2 \cdot s^{n_0}}{1-s} < \dfrac{\varepsilon/2}{1-s}$. For $s_0 = 1 - \varepsilon/2n_0 \le s < 1$ the first sum too is not greater than $\dfrac{\varepsilon/2}{1-s}$, and we have got (5.2) and (5.1).

We can rewrite formula (5.1) as

$$\lim_{s\to 1^-} P_{yy}(s) \cdot (1-s) = p_y. \tag{5.4}$$

We know that $F_{yy}(s) = 1 - \dfrac{1}{P_{yy}(s)}$ (formula (4.6)), $\dfrac{1 - F_{yy}(s)}{1-s} = \dfrac{1}{P_{yy}(s) \cdot (1-s)}$, so it follows from (5.4) that

$$\lim_{s \to 1^-} \frac{1 - F_{yy}(s)}{1-s} = \lim_{s \to 1^-} \frac{F_{yy}(1) - F_{yy}(s)}{1-s} = \frac{1}{p_y}, \tag{5.5}$$

where we are taking $1/0 = \infty$ ($F_{yy}(1) = \sum_{k=0}^{\infty} f_{yy}^{(k)} = 1$ because we supposed that the state y is recurrent).

The function F_{yy} is continuous in the interval $[s,\, 1]$ (because of uniform convergence of the power series defining this function) and differentiable in the interval $(s,\, 1)$; so the left-hand side of (5.5) can be written as $F'_{yy}(\tilde{s})$ for some $\tilde{s} \in (s,\, 1)$. We see that the limit of the derivative $F'_{yy}(\tilde{s})$ as $s \to 1$ is equal to $1/p_y$. As $s \to 1^-$, the intermediate point $\tilde{s}$ also goes to 1^-, so we have: $\lim_{\tilde{s} \to 1^-} F'_{yy}(\tilde{s}) = 1/p_y$ (this limit exists and is equal to $\sum_{k=0}^{\infty} k \cdot f_{yy}^{(k)}$). Finally we have $\sum_{k=0}^{\infty} k \cdot f_{yy}^{(k)} = E_y \tau_y = 1/p_y$, $p_y = 1/E_y \tau_y$.

We have introduced a classification of states $y \in X$: recurrent, transient. Let us introduce some more classification.

We say that a state y is *periodic* with period d, where d is an integer greater than 1 if $f_{yy}^{(k)} = 0$ except, possibly, k that are divisible by d:

$$k/d \text{ is not an integer } \Rightarrow\ f_{yy}^{(k)} = 0. \tag{5.6}$$

A state $y \in X$ is called *aperiodic* if it does not have any period > 1. (Out of the states mentioned in Problem $\boxed{1}$, the one in (b) is periodic with period 2, and the rest are aperiodic.)

Theorem 5.2. *Let y be an aperiodic state. Then the limit $\lim_{n \to \infty} p_{yy}^{(n)}$ exists and is equal to $1/E_y \tau_y$.*

I am going to prove first a theorem being a simpler particular case of Theorem 5.2:

Theorem 5.2′. *Let $f_{yy}^{(1)} > 0$. Then $\lim_{n \to \infty} p_{yy}^{(n)}$ exists and is equal to $1/E_y \tau_y$.* (Of course if $f_{yy}^{(1)} > 0$, the state y is aperiodic.)

Proof. It was mentioned that the statement is true for transient states, so we have to prove it only for recurrent states ($\sum_{k=1}^{\infty} f_{yy}^{(k)} = 1$).

There exist finite upper and lower limits

$$\overline{p} = \varlimsup_{n \to \infty} p_{yy}^{(n)}, \qquad \underline{p} = \varliminf_{n \to \infty} p_{yy}^{(n)}, \qquad 0 \le \underline{p} \le \overline{p} \le 1. \tag{5.7}$$

This means that for every $\varepsilon > 0$ there exists n_0 such that for all $n \ge n_0$

$$\underline{p} - \varepsilon < p_{yy}^{(n)} < \overline{p} + \varepsilon \tag{5.8}$$

and that there exist a sequence $n_i \to \infty$ such that

$$\lim_{i \to \infty} p_{yy}^{(n_i)} = \overline{p}, \tag{5.9}$$

and another sequence $n_i' \to \infty$,

$$\lim_{i\to\infty} p_{yy}^{(n_i')} = \underline{p}. \tag{5.10}$$

We are going to prove that for the sequence n_i also

$$\lim_{i\to\infty} p_{yy}^{(n_i-1)} = \overline{p}. \tag{5.11}$$

For this, we have to prove that for every $\varepsilon > 0$ there exists an i_0 such that for $i \geq i_0$ we have

$$p_{yy}^{n_i-1} > \overline{p} - \varepsilon \tag{5.12}$$

(as for $p_{yy}^{n_i-1} < \overline{p} + \varepsilon$, it follows from (5.8)).

Let us choose a natural k_0 so that $\sum_{k=k_0+1}^{\infty} f_{yy}^{(k)} < \varepsilon f_{yy}^{(1)}/3$, an n_0 so that the inequalities (5.8) hold with $\varepsilon f_{yy}^{(1)}/3$ instead of ε, and an i_0 so that $n_{i_0} \geq n_0 + k_0$ and that the inequality $p_{yy}^{(n_i)} > \overline{p} - \varepsilon f_{yy}^{(1)}/3$ holds for $i \geq i_0$.

We have (by the formula $p_{yy}^{(n)} = \sum_{k=1}^{n} f_{yy}^{(k)} p_{yy}^{(n-k)}$):

$$\begin{aligned}
\overline{p} - \varepsilon f_{yy}^{(1)}/3 < p_{yy}^{(n_i)} &= f_{yy}^{(1)} p^{(n-1)} + \sum_{k=2}^{k_0} f_{yy}^{(k)} p_{yy}^{(n_i-k)} + \sum_{k=k_0+1}^{n} f_{yy}^{(k)} p_{yy}^{(n_i-k)} \\
&< f_{yy}^{(1)} p^{(n_i-1)} + \sum_{k=2}^{k_0} f_{yy}^{(k)} (\overline{p} + \varepsilon f_{yy}^{(1)}/3) + \sum_{k=k_0+1}^{\infty} f_{yy}^{(k)} \\
&\leq f_{yy}^{(1)} p^{(n_i-1)} + (1 - f_{yy}^{(1)})(\overline{p} + \varepsilon f_{yy}^{(1)}/3) + \varepsilon f_{yy}^{(1)}/3 \\
&< f_{yy}^{(1)} p^{(n_i-1)} + \overline{p} + 2\varepsilon f_{yy}^{(1)}/3 - f_{yy}^{(1)} \overline{p},
\end{aligned} \tag{5.13}$$

$$f_{yy}^{(1)} p_{yy}^{(n_i-1)} > f_{yy}^{(1)} \overline{p} - \varepsilon f_{yy}^{(1)}, \tag{5.14}$$

$$p_{yy}^{(n_i-1)} > \overline{p} - \varepsilon. \tag{5.15}$$

Repeating this argument with $n_i - 1$ instead of n_i, we get that $pyy^{(n_i-2)} \to \overline{p}$ as $i \to \infty$, and generally, $p_{yy}^{(n_i-k)} \to \overline{p}$.

Just the same way, for the sequence n_i' such that $p_{yy}^{(n_i')} \to \underline{p}$ we get that $p_{yy}^{(n_i'-k)} \to \underline{p}$ for $k = 1, 2, 3, \ldots$.

Now let us introduce the sums (the remainders, or “tails”, of the series $\sum_{k=1}^{\infty} f_{yy}^{(k)}$)

$$r_k = \sum_{j=k+1}^{\infty} f_{yy}^{(j)}. \tag{5.16}$$

We have: $r_0 = 1$ (remember: we have excluded the case of transience); and $f_{yy}^{(k)} = r_{k-1} - r_k$. So we can rewrite the formula $p_{yy}^{(n)} = \sum_{k=1}^{n} f_{yy}^{(k)} p_{yy}^{(n-k)}$) as

$$\begin{aligned}
p_{yy}^{(n)} = r_0 p_{yy}^{(n)} &= \sum_{k=1}^{n} (r_{k-1} - r_k) p_{yy}^{(n-k)} \\
&= r_0 p_{yy}^{(n-1)} + r_1 p_{yy}^{(n-2)} + \ldots + r_{n-2} p_{yy}^{(1)} + r_{n-1} p_{yy}^{(0)} \\
&\qquad - r_1 p_{yy}^{(n-1)} - \ldots - r_{n-2} p_{yy}^{(2)} - r_{n-1} p_{yy}^{(1)} - r_n p_{yy}^{(0)}.
\end{aligned} \tag{5.17}$$

From this we get:

$$\sum_{k=0}^{n-1} r_k (p_{yy}^{(n-1-k)} - p_{yy}^{(n-k)}) - r_n p_{yy}^{(0)} = 0. \tag{5.18}$$

`In what follows some things were in` Lecture 5`, some will be in` Lecture 6`.`

My next statement is that for all n

$$\sum_{k=0}^{n} r_k p_{yy}^{(n-k)} = 1. \tag{5.19}$$

Indeed, for $n = 0$ this sum consists of one summand only: $r_0 p_{yy}^{(0)} = 1 \cdot 1 = 1$. And for $n > 0$ we have:

$$\sum_{k=0}^{n-1} r_k p_{yy}^{(n-1-k)} - \sum_{k=0}^{n} r_k p_{yy}^{(n-k)} = \sum_{k=0}^{n-1} r_k (p_{yy}^{(n-1-k)} - p_{yy}^{(n-k)}) - r_n p_{yy}^{(0)} = 0 \tag{5.20}$$

according to (5.18). So the difference of two neighboring sums of (5.19) is equal to 0, they are all equal to one another, and to the zeroth sum, which is equal to 1.

Now, the sum

$$\sum_{k=0}^{\infty} r_k = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} f_{yy}^{(j)} = \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} f_{yy}^{(j)} = \sum_{j=1}^{\infty} j \cdot f_{yy}^{(j)} = E_y \tau_y. \tag{5.21}$$

For every K we have:

$$1 = \sum_{k=0}^{n} r_k p_{yy}^{(n-k)} = \lim_{i \to \infty} \sum_{k=0}^{n_i} r_k p_{yy}^{(n_i-k)} \geq \lim_{i \to \infty} \sum_{k=0}^{K} r_k p_{yy}^{(n_i-k)} = \sum_{k=0}^{K} r_k \cdot \overline{p}. \tag{5.22}$$

From this we get that $\overline{p} \leq \dfrac{1}{\sum_{k=0}^{K} r_k}$ for every K, and the limit passage as $K \to \infty$ yields

$$\varlimsup_{n \to \infty} p_{yy}^{(n)} \leq \frac{1}{\sum_{k=0}^{\infty} r_k} = \frac{1}{E_y \tau_y}. \tag{5.23}$$

This solves the problem in the case of $E_y \tau_y = \infty$: $\lim_{n \to \infty} p_{yy}^{(n)} = 0$. In the case of $E_y \tau_y < \infty$ we have to take care of the lower limit as well, and consider the sequence n_i'.

In the case of a finite expectation $E_y \tau_y$ we have for an arbitrary K:

$$1 = \lim_{i \to \infty} \sum_{k=0}^{n_i'} r_k p_{yy}^{(n_i'-k)} \leq \lim_{i \to \infty} \sum_{k=0}^{K-1} r_k p_{yy}^{(n_i'-k)} + \sum_{k=K}^{\infty} r_k = \sum_{k=0}^{K-1} r_k \cdot \underline{p} + \sum_{k=K}^{\infty} r_k. \tag{5.24}$$

Taking $K \to \infty$, we obtain:

$$1 \le \underline{p} \cdot \sum_{k=0}^{\infty} r_k, \qquad \underline{p} \ge \frac{1}{E_y \tau_y}. \tag{5.25}$$

From this, together with (5.23) and $\underline{p} \le \overline{p}$, follows $\underline{\lim}_{n\to\infty} p_{yy}^{(n)} = \overline{\lim}_{n\to\infty} p_{yy}^{(n)} = \lim_{n\to\infty} p_{yy}^{(n)} = 1/E_y \tau_y$.

Theorem 5.2′ is proved.

Now to the **proof** of Theorem 5.2.

Suppose that $f_{yy}^{(k)} > 0$ for $k = k_1$, ..., k_r (...,): a finite or an infinite number of k's. Exactly the same way as in the proof of Theorem 5.2′ we prove that $p_{yy}^{(n_i - k_j)} \to \overline{p}$ as $i \to \infty$, then that $p_{yy}^{(n_i - k_{j_1} - k_{j_2})} \to \overline{p}$; etc.: the original sequence n_i from which some number or k's with positive $f_{yy}^{(k)}$ is subtracted.

Aperiodicity means that the largest common divisor of k_1, ..., k_r (,...) is equal to 1. Of course if the largest common divisor of an infinite set of integers is equal to 1, we can choose from them finitely many with the largest common divisor 1. And here we are going to use a lemma belonging to the number theory (and as usual I won't give its proof, because it is not about probability theory):

Lemma 5.1. *If the largest common divisor of positive integers* k_1, ..., k_r *is equal to* 1, *then there exists an integer* N *such that every integer* $k \ge N$ *can be represented as the sum of our* k_1, ..., k_r *repeated, perhaps, several times*: $k = m_1 \cdot k_1 + ... + m_r \cdot k_r$, *where* m_1, ..., m_r *are nonnegative integers* (in fact, we can take $N = (k_1 - 1) \cdot ... \cdot (k_r - 1)$).

Now the same way as in the proof of Theorem 5.2′ we have that $\lim_{i\to\infty} p^{(n_i - k)} = \overline{p}$, $\lim_{i\to\infty} p^{(n'_i - k)} = \underline{p}$ for $k \ge N$. Using the equalities $1 = \sum_{k=0}^{n_i - N} r_k p_{yy}^{n_i - N - k}$, $1 = \sum_{k=0}^{n'_i - N} r_k p_{yy}^{n'_i - N - k}$ the same way as we used them with n_i and n'_i instead of $n_i - N$, $n'_i - N$, we get the proof.

Theorem 5.3. *If the state* y *is periodic with its largest period* d, *then* $p_{yy}^{(n)} = 0$ *for* n *not divisible by* d, *and* $\lim_{k\to\infty} p_{yy}^{(dk)} = d/E_y \tau_y$.

Proof. Introduce a new Markov chain $\tilde{\xi}_n = \xi_{dn}$. Since ξ_n can return to the state y only after a time that is a multiple of d, the returning time $\tilde{\tau}_y$ for our new chain is equal to τ_y / d; and the state y is aperiodic for our new chain (because d was *the largest* period). Applying Theorem 5.2 to the chain $\tilde{\xi}_n$, we get: $\lim_{k\to\infty} p_{yy}^{(dk)} = \lim_{k\to\infty} \tilde{p}_{yy}^{(k)} = 1/E_y \tilde{\tau}_y = d/E_y \tau_y$.