MATH 777 MARKOV STOCHASTIC PROCESSES SPRING 2009

Problems.

1 Can you invent a discrete Markov chain so that for some state y the first return time to y has the following distribution?

(a) $P_y\{\tau_y = 1\} = f_{yy}^{(1)} = 1$; (b) $f_{yy}^{(2)} = 1$; (c) $f_{yy}^{(1)} = f_{yy}^{(2)} = 1/2$; (d) $f_{yy}^{(1)} = 1/3$, $f_{yy}^{(2)} = 2/3$; (e) $f_{yy}^{(1)} = 1/2$, $P_y\{\tau_y = \infty\} = 1/2$; (f) $f_{yy}^{(1)} = f_{yy}^{(2)} = f_{yy}^{(3)} = 1/3$; (g) $f_{yy}^{(k)} = \dfrac{1}{k(k+1)}$, $k = 1, 2, 3, \ldots$.

2 Let me introduce the random variables τ_y^k (the time of the k-th return to the state y) by $\tau_y^1 = \tau_y$, $\tau_y^k = \min\{n > \tau_y^{k-1} :\ \xi_n = y\}$ (and, of course, by definition $\tau_y^k = \infty$ if there are no such n).

Prove that for a recurrent state all τ_y^k are finite almost surely with respect to the probability P_y; and for a transient y, P_y-almost surely we return to the state y only finitely many times: almost surely all τ_y^k starting with some (random) k are equal to $+\infty$.

3 Give an example of an integer-valued random variable η not equal to a constant almost surely, for which the characteristic function is equal to 1 in absolute value at three points in the interval $(-\pi, \pi]$.

4 Check that the (2π)-periodic function $f(t) = 1 - |\sin(t/2)|$ is a characteristic function of a discrete distribution. For this, find its Fourier coefficients p_j and check that they are nonnegative, $\sum_{j=-\infty}^{\infty} p_j = 1$.

5 Does the series $\sum_{j=-\infty}^{\infty} |j| \cdot p_j$ converge or diverge?

6 Is $\sum_{n=0}^{\infty} \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(t)^n \, dt = \infty$ or $< \infty$? Are the states in the Markov chain with transition matrix (2008.23 – 24.23) with p_j of Problem **4** recurrent or transient?

7 $f_{yy}^{(k)} = 0$ for all k that are not multiples of d if and only if $p_{yy}^{(n)} = 0$ for all n that are not multiples of d.

The deadline for Problems **1**–**7** is Feb. 6.

8 Let the transition matrix of a Markov chain on the space $X = \{1, 2, ..., 10\}$ be

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the equivalence classes C_i. Which of them are recurrent and which transient? For each recurrent class find out whether it is periodic or aperiodic; in the periodic case, find the subclasses C_{ij}. For all states $y \in X$, find $E_y \tau_y$, $\overline{\lim}_{n\to\infty} p_{yy}^{(n)}$, and $\underline{\lim}_{n\to\infty} p_{yy}^{(n)}$.

In Problems **9**–**14**, let all states of a Markov chain form one equivalence class: $x \sim y$ for all $x, y \in X$.

We know (see Lecture Note # 7) that the vector $\boldsymbol{p}$ of limiting probabilities (if they exist) is a (left) eigenvector of the transition matrix P with eigenvalue 1: $\boldsymbol{p} \cdot P = \boldsymbol{p}$; and either $\boldsymbol{p} = \mathbf{0}$, or $\sum_y p_y = 1$ (in the vector-matrix form: $\boldsymbol{p} \cdot \mathbf{1} = 1$, where $\mathbf{1}$ is the column vector with all components equal to 1). Also we used the equality $\boldsymbol{p} \cdot P^n = \boldsymbol{p}$.

9 Prove that in the ergodic case the nonnegative solution of the problem $\boldsymbol{p} \cdot P = \boldsymbol{p}$, $\boldsymbol{p} \cdot \mathbf{1} = 1$ is unique: if $\boldsymbol{q}$ is an arbitrary left eigenvector with components $q_y \geq 0$, $\boldsymbol{q} \cdot P = \boldsymbol{q}$, $\boldsymbol{q} \cdot \mathbf{1} = 1$, then $\boldsymbol{q} = \boldsymbol{p}$.

10 Prove that if $\boldsymbol{q} = (q_y)_{y \in X}$, $q_y \geq 0$, $\boldsymbol{q} \cdot P = \boldsymbol{q}$, $\boldsymbol{q} \cdot \mathbf{1} = 1$, then it is impossible that $\lim_{n\to\infty} p_{xy}^{(n)} = 0$.

11 Prove that in the d-periodic case the row vector $\boldsymbol{p}$ with components $p_y = \lim_{n\to\infty} \dfrac{p_{xy}^{(n)} + p_{xy}^{(n+1)} + ... + p_{xy}^{(n+d-1)}}{d}$ is the solution of $\boldsymbol{p} \cdot P = \boldsymbol{p}$, and $p_y = 1/E_y \tau_y$.

12 Prove that in the case of d-periodic states with $E_y \tau_y < \infty$ for all $y \in X$ the nonnegative solution of $\boldsymbol{p} \cdot P = \boldsymbol{p}$, $\boldsymbol{p} \cdot \mathbf{1} = 1$ is unique.

13 Prove in the case of d-periodic states that if $\boldsymbol{q} = (q_y)_{y \in X}$, $q_y \geq 0$, $\boldsymbol{q} \cdot P = \boldsymbol{q}$, $\boldsymbol{q} \cdot \mathbf{1} = 1$, then $q_y = 1/E_y \tau_y$ (not that $q_y = \lim_{n\to\infty} p_{xy}^{(n)}$).

14 Prove that if $\boldsymbol{q} \cdot P = \boldsymbol{q}$, its components $q_y \in [0, \infty)$, $\sum_y q_y = \infty$ (i.e. $\boldsymbol{q} \cdot \mathbf{1} = \infty$), then $\lim_{n\to\infty} p_{xy}^{(n)} = 0$.

15 Let $X = \mathbb{Z}^1$, $p_{x,\,x-1} = 1/3$, $p_{x,\,x+1} = 2/3$ for $x < 0$, $p_{x,\,x-1} = 2/3$, $p_{x,\,x+1} = 1/3$ for $x > 0$, and $p_{0,\,-1} = 1/2$, $p_{00} = 1/6$, $p_{01} = 1/3$. Do the states of this chain form one equivalence class? Are they periodic? Are they recurrent or transient? Do the limits $\lim_{n\to\infty} p_{xy}^{(n)}$ exist? If they exist, what are they equal to? Find $E_y\tau_y$ for all $y \in \mathbb{Z}^1$.

16 Let $X = \mathbb{Z}^1$, $p_{x,\,x-1} = 2/3$, $p_{x,\,x+1} = 1/3$ for $x < 0$, $p_{x,\,x-1} = 1/3$, $p_{x,\,x+1} = 2/3$ for $x > 0$, and $p_{0,\,-1} = 1/2$, $p_{00} = 1/6$, $p_{01} = 1/3$. Do the states of this chain form one equivalence class? Are they periodic? Are they recurrent or transient? Do the limits $\lim_{n\to\infty} p_{xy}^{(n)}$ exist? If they exist, what are they equal to? Find $E_y\tau_y$ for all $y \in \mathbb{Z}^1$.

17 Let $X = \mathbb{Z}^1$, $p_{x,\,x-1} = p_{x,\,x+1} = 1/2$ for $x < 0$, $p_{x,\,x-1} = p_{x,\,x+1} = 1/2$ for $x > 0$, and $p_{0,\,-1} = 1/2$, $p_{00} = 1/6$, $p_{01} = 1/3$. Do the states of this chain form one equivalence class? Are they periodic? Are they recurrent or transient? Do the limits $\lim_{n\to\infty} p_{xy}^{(n)}$ exist? If they exist, what are they equal to? Find $E_y\tau_y$ for all $y \in \mathbb{Z}^1$.

The deadline for Problems **8**–**17** is Feb. 13.

18 For the simplest random walk on $\mathbb{Z}^1$, what is the expectation, starting from the point $\xi_0 = 1$, of the first time of reaching the point 0? What is the expectation of the same random variable starting from the point $\xi_0 = -1$ (i. e., with respect to the probability P_{-1})?

19 For the discrete Markov chain on the space $X = \{0, 1, 2, 3, ...\}$ with one-step transition probabilities $p_{01} = 1$, $p_{x,\,x+1} = p_{x,\,x-1} = 1/2$, $x > 0$ (the rest 0), is the state 0 recurrent. If yes, does $\lim_{n\to\infty} p_{xy}^{(n)}$ exist? If yes, what is this limit equal to?

20 For the same Markov chain, what is the expectation, starting from the point $\xi_0 = 1$, of the first time of reaching the point 0?

21 Prove or disprove: If all exit rates v_x are positive, a state y for a continuous-time Markov chain ξ_t is transient if and only if it is transient for the discrete Markov chain $\eta_n = \xi_{\tau_n}$.

22 Prove or disprove: If $P_x\{\lim_{n\to\infty} \tau_n = \infty\} = 1$ and all exit rates v_x are positive, $\lim_{t\to\infty} p(t,\, y,\, y)$ exists and is positive if and only if $\lim_{n\to\infty} \pi_{yy}^{(n)}$ exists and is positive, where $\pi_{xy}^{(n)}$ are the n-step transition probabilities of the discrete Markov chain $\eta_n = \xi_{\tau_n}$.

23 Prove or disprove: If $P_x\{\lim_{n\to\infty} \tau_n = \infty\} = 1$ and all exit rates v_x are positive, $\lim_{t\to\infty} p(t,\, y,\, y)$ exists and is positive if and only if $\overline{\lim}_{n\to\infty} \pi_{yy}^{(n)} > 0$.

24 Consider a gas station with two pumps and an unrestricted queue, with independent exponential interarrival and service times with parameters, respectively, λ and μ.

For $\lambda = 3$, $\mu = 2$, is the continuous-time Markov chain on $X = \{0, 1, 2, 3, ...\}$ describing the number of customers in the system transient? Find $\lim_{t\to\infty} p(t,\, x,\, y)$.

25 The same for $\lambda = 3$, $\mu = 1$.

26 The same for $\lambda = 2$, $\mu = 1$.

27 For the idealized gas station with *infinitely many pumps*, what are the exit and transition rates a_{xy}? What are the limits $\lim_{t\to\infty} p(t,\, x,\, y)$?

28 Let us consider a continuous-time Markov chain on $X = \{0, 1, 2, 3, ...\}$ with exit rates $v_0 = 1$, $v_x = 2x$ for $x > 0$ and transition rates $a_{01} = 1$, $a_{x,\,x+1} = a_{x,\,x-1} = x$ for $x > 0$.

Is this continuous-time Markov chain transient?

Is $\lim_{n\to\infty} \tau_n = \infty$ almost surely?

What is $\lim_{t\to\infty} p(t, x, y)$ equal to?

Hint for Problems **18**–**20**, **24**–**27**: First find the general solution of the system $\boldsymbol{q} \cdot A = \mathbf{0}$, and all *nonnegative* solutions of this system; use the analogues of Problems **9**–**12**, **14** that are given, without proofs, on p. 4 of the Lecture Note # 13.

Deadline for Problems up to **28** is March 20.

29 Let ξ_t, $t \geq 0$, be a one-dimensional Wiener process; let P_x be the probabilities evaluated under the assumption that this process starts from the point x at time $t = 0$. Let $\tilde{\xi}_t = |\xi_t|$ (the Wiener process reflected at a mirror placed at the point $x = 0$). I am planning to ask you to prove that $\tilde{\xi}_t$ is a Markov process; but first this question: *If* $\tilde{\xi}_t$, considered with respect to the probabilities P_x, $x \geq 0$, is a Markov process, what is its transition function $\tilde{P}(t, x, C)$? Does this function have a density (transition density) with respect to the Lebesgue measure?

30 Check that the transition function $\tilde{P}(t, x, C)$ that you found solving the previous problem satisfies everything that we require from a transition function (on the space $X = [0, \infty)$).

31 Check that $\tilde{\xi}_t$, $t \geq 0$, is a Markov process with the transition function $\tilde{P}(t, x, C)$ that you found.

32 Let ξ_t° be defined as ξ_t if $\tau_0 > t$, and as 0 if $\tau_0 \leq t$ (ξ_t° is the Wiener process stopped at its first reaching the point 0; make a picture of a trajectory of ξ_t°). Using what we found in Example 19.1, answer this question: If ξ_t° is a Markov process with respect to the probabilities P_x, $x \geq 0$, what is its transition function $P^\circ(t, x, C)$? Does this function have a density with respect to the Lebesgue measure?

33 Check that the transition function $P^\circ(t, x, C)$ that you found solving the previous problem satisfies everything that we require from a transition function (on the space $X = [0, \infty)$).

34 Check that ξ_t°, $t \geq 0$, is a Markov process with the transition function $P^\circ(t, x, C)$ that you found.

32 − 34 It turned out that, while Problems **29**–**31** are solved quite naturally in the order they are given, it's not so for problems **32**–**34**. So I propose a different order:

Of course the transition function $P^\circ(t, x, C) = P_x\{\xi_t^\circ \in C\}$. Using the fact that for ω such that $\tau_0(\omega) > t$ we have $\xi_{t+s}^\circ(\omega) = \xi_s^\circ(\theta_t\omega)$, prove that $P_x\{\xi_{t+s}^\circ \in C \| \mathcal{F}_{\leq t}\} = P^\circ(s, \xi_t^\circ, C)$ (consider two cases: $\tau_0 > t$, and $\tau_0 \leq t$). This already means that ξ_t° is a Markov process with transition function P° (Problem **34**). Then you check that that transition function satisfies all properties it should (Problem **33**). And then you write the explicit expression for the transition function (Problem **32**).

35 Does the one-dimensional Wiener process satisfy the condition of Theorem 20.3? The same question for the processes of Problems **29**–**34**.

36 Let ξ_t, $t \ge 0$, be the one-dimensional Wiener process; $\eta_t = e^{\xi_t}$.

Prove that η_t, $t \ge 0$, is also a Markov process on the half-line $(0, \infty)$ (find its transition function $P^\eta(t, x, C)$ first).

Is the transition function of the process η_t described by a transition density? If yes, find this density.

Is the process η_t uniformly continuous in probability? Does its transition function satisfy the condition $\sup_{x \in (0, \infty)} P^\eta\big(h, x, \{y\colon \mathrm{dist}(y, x) \ge \varepsilon\}\big) = o(h)$ as $h \to 0^+$?

37 Prove that a continuous-time Markov chain with exit rates v_x bounded is uniformly continuous in probability.

38 Can a continuous-time Markov chain with *un*bounded exit rates v_x be uniformly continuous in probability?

39 Let ξ_t, $t \ge 0$, be a continuous-time Markov chain with the matrix A of rates given by $A = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$. Solving differential equations, find the transition probabilities $p(t, x, y)$ of this chain.

40 Let ξ_t, $t \ge 0$, be a continuous-time Markov chain on $X = \{0, 1, 2, 3, ...\}$ with exit and transition rates $v_{2k} = v_{2k+1} = a_{2k,\, 2k+1} = a_{2k+1,\, 2k} = k$. Find $\alpha = \sup_{x \in X,\, h \le h_0} P(h, x, X \setminus \{x\})$.

Do almost all trajectories of this continuous-time Markov chain have left-hand limits at every $t > 0$?

41 Let ξ_t, $t \ge 0$, be a continuous-time Markov chain on $X = \{0, 1, 2, 3, ...\}$ with exit and transition rates $v_x = a_{x,\, x+1} = 2^x$. Do almost all trajectories of this continuous-time Markov chain have left-hand limits at every $t > 0$?

42 Check that the function $p(t, x, y) = \dfrac{\pi^{-1} t}{t^2 + (y - x)^2}$, $t > 0$, $x, y \in \mathbb{R}^1$, satisfies all requirements that the transition density of a Markov process on $(-\infty, \infty)$ must satisfy (i.e. that $P(t, x, C) = \displaystyle\int_C p(t, x, y)\, dy$ is a Markov transition function). Does there exist a Markov process with right-continuous trajectories with this transition density?

Deadline for Problems up to **42** is Apr. 15.

43 Let P^t be the semigroup of linear operators on $\mathbf{B}(-\infty, \infty)$ associated with the one-dimensional Wiener process ξ_t. For a function $f \in \mathbf{B}[0, \infty)$ let us denote with $\hat{f}(x)$ the function f extended to the whole real line as an even function: $\hat{f}(x) = f(|x|)$. Let us define the operators $\hat{P}^t$ acting on the space $\mathbf{B}[0, \infty)$ by $\hat{P}^t f(x) = P^t \hat{f}(x)$, $x \in [0, \infty)$.

Check that $\hat{P}^t$, $t \geq 0$, is a one-parameter semigroup associated with a Markov transition function.

44 Let P^t be the semigroup of linear operators on $\mathbf{B}(-\infty, \infty)$ associated with the one-dimensional Wiener process ξ_t. For a function $f \in \mathbf{B}[0, \infty)$ let us denote with $\tilde{f}(x)$ the function f extended to the whole real line as a function symmetric with respect to the point $\big(0, f(0)\big)$: $\tilde{f}(x) = 2f(0) - f(-x)$ (make a picture). Let us define the operators $\tilde{P}^t$ acting on the space $\mathbf{B}[0, \infty)$ by $\tilde{P}^t f(x) = P^t \tilde{f}(x)$, $x \in [0, \infty)$.

Check that $\tilde{P}^t$, $t \geq 0$, is a one-parameter semigroup associated with a Markov transition function.

You can think also about what Markov processes correspond to these semigroups; but we don't need it now.

45 Let the semigroup P^t corresponding to a Markov process ξ_t in a metric space X take the space $\mathbf{C}(X)$ to $\mathbf{C}(X)$, and let P^t be continuous on $\mathbf{C}(X)$. Let x_0 be a point such that $Af(x_0) = 0$ for every $f \in D_A$.

Using the fact that $\dfrac{d}{dt} P^t f = P^t A f = A P^t f$ for $f \in D_A$, prove that $P^t f(x_0) = f(x_0)$ for every $f \in D_A$.

What equality did you use: $\dfrac{d}{dt} P^t f = P^t A f$ or $\dfrac{d}{dt} P^t f = A P^t f$?

46 Under the same conditions, prove $P^t f(x_0) = f(x_0)$ for every $f \in \mathbf{C}(X)$.

It follows from this that $P(t, x_0, \{x_0\}) = 1$: the process almost surely does not leave the point x_0 if it starts from it (or, by the strong Markov property, if it ever gets to the point x_0).

47 Let $X = [0, \infty]$, $D_A = \{f \in \mathbf{C}^2[0, \infty] : c \cdot \frac{1}{2} f''(0) = b \cdot f'(0)\}$, where c and b are nonnegative constants, $b + c > 0$; $Af(x) = \frac{1}{2} f''(x)$.

The operator A is the infinitesimal operator of a continuous semigroup on $\mathbf{C}[0, \infty]$ corresponding to a Markov process (you are not asked to check this).

Prove that this process can be taken with continuous trajectories.

48 Prove that for a continuous function $f(\boldsymbol{x})$ that doesn't grow faster than exponentially: $|f(\boldsymbol{x})| \leq C_1 e^{C_2|\boldsymbol{x}|}$, $P^t f(\boldsymbol{x}) = \int_{\mathbb{R}^d} p(t,\, \boldsymbol{x},\, \boldsymbol{y}) \cdot f(\boldsymbol{y})\, d\boldsymbol{y}$ is defined, and we have $\lim_{t \to 0+} P^t f(\boldsymbol{x}) = f(\boldsymbol{x})$ ($p(t,\, \boldsymbol{x},\, \boldsymbol{y})$ is the transition density of the diffusion process; it satisfies the forward Kolmogorov equation (27.12) and the estimates (27.13), (27.14)).

49 Prove that for a twice continuously differentiable function $f(\boldsymbol{x})$ growing, together with its first and second derivatives, not faster than exponentially: $|f(\boldsymbol{x})|, \left|\frac{\partial f}{\partial x_i}\right|, \left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| \leq C_1 e^{C_2|\boldsymbol{x}|}$, we have: $\frac{\partial}{\partial t} P^t f(\boldsymbol{x}) = P^t L f(\boldsymbol{x})$ for $t > 0$ (prove this by integrating by parts), and $P^t f(\boldsymbol{x}) - f(\boldsymbol{x}) = \int_0^t P^u L f(\boldsymbol{x})\, du$ (where $P^t f(\boldsymbol{x}) = \int_{\mathbb{R}^d} p(t,\, \boldsymbol{x},\, \boldsymbol{y}) \cdot f(\boldsymbol{y})\, d\boldsymbol{y}$).

50 Check that the function $f(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|}$ in $\mathbb{R}^3$ satisfies the equation $\Delta f(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \neq \mathbf{0}$. Check that for the three-dimensional Wiener process $E_{\boldsymbol{x}} f(\boldsymbol{\xi}_t) < \infty$ for $t > 0$.

Find $\lim_{t \to \infty} E_{\boldsymbol{x}} f(\boldsymbol{\xi}_t)$.

51 Is the random function $\eta_t = f(\boldsymbol{\xi}_t)$ a martingale?

52 Prove that for the function (30.2) for every $x \in (-\infty,\, \infty)$ we have:

$$\lim_{x^1 \to x,\, x^2 \to 0+} u(x^1,\, x^2) = \varphi(x).$$

We were considering what can be done using the fact that a function f applied to a Markov process, minus the integral (or the sum, see Example 27.1) of the values of some other function g at previous times, is a Martingale. For arbitrary Markov process, the result (Example 27.2) required that f should belong to the domain of definition of the infinitesimal operator of the corresponding semigroup – and so f and $g = Af$ were required to be bounded. In the particular case of diffusion processes, using some sophisticated results about partial differential equations, we were able to have the same fact for the function f growing, together with its derivatives, not very fast; and we applied this to studying the Wiener process.

I am going to give some problems in which the result of Example 27.2 (and 27.1) are applied to some different Markov processes (and discrete-time Markov chains).

53 Let ξ_t, $t \geq 0$, be the one-dimensional Wiener process. Prove that almost surely the total time spent by this process at a point x_0, which is the one-dimensional Lebesgue measure $\lambda_1\{t : \xi_t(\omega) = x_0\}$, is equal to 0. To do this, write $E_x \lambda_1\{t : \xi_t(\omega) = x_0\} = E_x \int_0^\infty I_{\{x_0\}}(\xi_t)\, dt = \int_\Omega \Big[\int_0^\infty I_{\{x_0\}}\big(\xi_t(\omega)\big)\, dt\Big]\, P_x(d\omega)$, and use Fubini's Theorem.

54 Let ξ_t, $t \geq 0$, be the Markov process with continuous trajectories corresponding to the infinitesimal operator A of Problem **47** with $b = 1$. For $a > 0$, $k > 0$, evaluate the expectation $u_k(x) = E_x \int_0^{\tau_{[0,\, a)}} e^{-k\xi_s}\, ds$, where $\tau_{[0,\, a)}$ is the first time that ξ_t leaves the open set $[0, a)$.

To do this, find the solution of the equation $\frac{1}{2}\, u_k''(x) = -\, e^{-kx}$, $0 \leq x \leq a$, with the boundary conditions $c \cdot \frac{1}{2}\, u_k''(0) = u_k'(0)$, $u_k(a) = 0$ (checking also that this function can be extended as a function belonging to D_A to the whole extended half-line).

55 Taking $k \to \infty$ in the solution found in the previous problem, find the expectation of the time spent at 0 before leaving the interval $[0, a)$: $E_x \int_0^{\tau_{[0,\, a)}} I_{\{0\}}(\xi_t)\, dt$.

The deadline for Problems **43**–**55** is Apr. 27. For the rest of the problems, May 4.

56 Let $X = \mathbb{Z}^1$; for integers $a < b$, let us denote $(a, b) = \{x \in \mathbb{Z}^1 : a < x < b\} = \{a+1, a+2, ..., b-1\}$. Let ξ_n, $n = 0, 1, 2, 3, ...$, be the simplest random walk on $\mathbb{Z}^1$ (with probability 1/2 a step to the left, and a step to the right with the same probability).

Let $\tau_{(a,\, b)}$ be the first time at which ξ_n leaves the interval (a, b). Using the martingale of Example 27.3, prove that $E_x \tau_{(a,\, b)} = m(x) = (x-a)(b-x)$ for $a \leq x \leq b$.

57 Let $X = \mathbb{Z}^1$; for integers $a < b$, let us denote $(a, b) = \{x \in \mathbb{Z}^1 : a < x < b\} = \{a+1, a+2, ..., b-1\}$. Let ξ_n, $n = 0, 1, 2, 3, ...$, be the Markov chain with one-step transition probabilities $p_{x,\, x-1} = p_{x,\, x+1} = p_{x,\, x+2} = 1/3$ (the rest 0).

Prove that $E_x \tau_{(a,\, b)} < \infty$.

58 For the chain of the previous problem starting from a point $x \in (a, b)$, the exit point $\xi_{\tau_{(a,\,b)}}$ can only be equal to a, or b, or $b+1$.

Solving the equation $(P - I)u(x) = 0$ with appropriate "boundary conditions", find the probabilities $P_x\{\xi_{\tau_{(a,\,b)}} = a\}$, $P_x\{\xi_{\tau_{(a,\,b)}} = b\}$, and $P_x\{\xi_{\tau_{(a,\,b)}} = b+1\}$.

To find the general solution of $(P - I)u(x) = 0$, find three (possibly unbounded) linearly independent solutions of this equation of the form $u(x) = \gamma^x$.

59 Let $X = (-\infty, a]$, $a > 0$; the operator A is defined on all functions $f \in \mathbf{B}(X, \mathcal{B}_X)$ by $Af(x) = \int_{-\infty}^{x} e^{y-x} \cdot [f(y) - f(x)]\, dy$. Let ξ_t be the right-continuous pure-jump Markov process with infinitesimal operator A (see Example 23.1). Let $\tau = \tau_{(0,\,a]}$ be the first time that this process leaves the interval $(0, a]$.

Considering the function

$$f(x) = \begin{cases} 0, & x \le 0, \\ x+1, & 0 < x \le a, \end{cases}$$

and $Af(x)$, prove that $E_x\tau < \infty$.

60 For the same process, considering a function $f \in \mathbf{B}(X, \mathcal{B}_X)$ defined arbitrarily for $x \le 0$, and as the constant $\overline{f} = \int_{-\infty}^{0} e^{y} \cdot f(y)\, dy$ for $x \in (0, a]$, find the distribution (with respect to the probability P_x, $x \in (0, a]$) of the random variable ξ_τ.

61 Let W_t^1, W_t^2, $t \ge t_0$, be two independent Wiener processes (as, e. g., two different coordinates of a multidimensional Wiener process. Their being independent means that for every finite collection of time moments $t_0 < t_1 < t_2 < \ldots < t_n$ the random vectors $(W_{t_0}^1, W_{t_1}^1, W_{t_2}^1, \ldots, W_{t_n}^1)$ and $(W_{t_0}^2, W_{t_1}^2, W_{t_2}^2, \ldots, W_{t_n}^2)$ are independent). For a partition $\mathfrak{T}$ of the interval from a to b with partition points $t_0 = a < t_1 < t_2 < \ldots < t_n = b$, take

$$\Sigma_{\mathfrak{T}}^{1,\,2} = \sum_{i=1}^{n} (W_{t_i}^1 - W_{t_{i-1}}^1) \cdot (W_{t_i}^2 - W_{t_{i-1}}^2).$$

Prove that there exists a mean-square limit

$$\operatorname*{l.i.m.}_{\max_{1 \le t \le n}(t_i - t_{i-1}) \to 0} \Sigma_{\mathfrak{T}}^{1,\,2}.$$

Is this limit a constant or a non-constant random variable?

If a constant, what is it equal to? If non-constant, what can you say about its distribution?

62 Let $\eta_t = e^{-\alpha t} e^{\sigma W_t}$, where W_t is a Wiener process. For a given constant $\sigma \ne 0$, can we choose the coefficient α so that η_t is a martingale?

63 Let the stochastic process ξ_t, $t \ge 0$, be defined as $\xi_t = \sin W_t$, where W_t is a Wiener process starting at $t = 0$ from a point $x_0 \in \mathbb{R}^1$ (i. e., $W_0 = x_0$). This process takes only values in the interval $[-1, 1]$.

Prove or disprove that ξ_t is a solution of the stochastic differential equation

$$d\xi_t = \sqrt{1 - \xi_t^2}\, dW_t - \tfrac{1}{2}\, \xi_t\, dt.$$

64 Applying Itô's formula to the random function (35.15), prove that for

$$\eta_t = v_0 e^{-ct} + \sigma \int_0^t e^{-c(t-s)}\, dW_s, \qquad t \ge 0,$$

we have:

$$d\eta_t = -c\eta_t\, dt + \sigma\, dW_t.$$

65 Let $F(x, y)$ be a bounded $(\mathcal{X} \times \mathcal{Y})$-measurable function on the space $(X \times Y,\, \mathcal{X} \times \mathcal{Y})$. Let ξ be a random variable with values in the space $(X, \mathcal{X})$, measurable with respect to a σ-algebra $\mathcal{A}$; let η be a random variable with values in $(Y, \mathcal{Y})$ that is independent with the σ-algebra $\mathcal{A}$. Let $G(x) = EF(x, \eta)$.

Prove that $E\big(F(\xi, \eta)\big\|\mathcal{A}\big) = G(\xi)$ (almost surely, of course).

First we prove that $G(x)$ is $\mathcal{X}$-measurable; after that the statement about the conditional expectation reduces to some equality with unconditional expectations.

The expectation of an arbitrary random variable is defined as the limit of expectations of linear combinations of indicators; using this, we can reduce our problem ro proving the same for indicators $I_C(x, y)$ of sets $C \in \mathcal{X} \times \mathcal{Y}$.

And $\mathcal{X} \times \mathcal{Y}$, as we know, is the σ-algebra generated by "rectangles" $A \times B$, $A \in \mathcal{X}$, $B \in \mathcal{Y}$. Using this we can reduce our statement to proving the same for $F(x, y) = I_A(x) \cdot I_B(y)$.

66 Let us consider the differential operator $Lf(x) = \dfrac{1}{2}\dfrac{d}{dx}\big(\ln(1 + x^2) \cdot f'(x)\big)$. Write a stochastic equation whose solution is a diffusion process with L as its generating operator. Do the coefficients of this equation satisfy the conditions for existence and uniqueness? Does the differential operator L satisfy the conditions of Lecture 21 allowing to treat it using the results on parabolic differential equations?

67 For the one-dimensional Wiener process W_t, find the expectation $E_x e^{c\tau_{(a,\,b)}}$, where c is a negative constant and $\tau_{(a,\,b)}$ the first time at which the Wiener process leaves the interval (a, b).

68 Let $(a, b) = (-1, 1)$, $\tau = \tau_{(-1,\,1)}$ the first time at which the Wiener process leaves this interval. Find the expectation $E_{0.1} e^{15\tau}$ with accuracy up to 0.01 (if the expectation is infinite, explain it).