Lecture note 7 − 8. Stochastic integral equations. Existence and uniqueness.

I am including in every lecture, and in every lecture note some things whose place should really be in some of the previous lectures, but that were not mentioned them, or stressed enough. Here it is, for this lecture note.

I have mentioned the Schwarz inequality:

$$|E(XY)| \leq \sqrt{E(X^2) \cdot E(Y^2)}. \tag{7-8.1}$$

From this we have:

$$\begin{aligned} E\big((X+Y)^2\big) &= E(X^2) + 2E(XY) + E(Y^2) \\ &\leq E(X^2) + 2\sqrt{E(X^2) \cdot E(Y^2)} + E(Y^2) = \Big[\sqrt{E(X^2)} + \sqrt{E(Y^2)}\Big]^2, \end{aligned} \tag{7-8.2}$$

$$E\Big(\big(\sum_{i=1}^n X_i\big)^2\Big) = \sum_{i,\,j=1}^n E(X_i X_j) \leq \sum_{i,\,j=1}^n \sqrt{E(X_i^2)} \cdot \sqrt{E(X_j^2)} = \Big[\sum_{i=1}^n \sqrt{E(X_i^2)}\Big]^2. \tag{7-8.3}$$

We can also write:

$$E\big((X+Y)^2\big) \leq 2E(X^2) + 2E(Y^2). \tag{7-8.4}$$

If we introduce the notation

$$\|X\|_2 = \sqrt{E(X^2)} \tag{7-8.5}$$

(read: "the norm of X"), we can write:

$$\|X+Y\|_2 \leq \|X\|_2 + \|Y\|_2, \tag{7-8.6}$$

$$\Big\|\sum_{i=1}^n X_i\Big\|_2 \leq \sum_{i=1}^n \|X_i\|_2. \tag{7-8.7}$$

If X_1, X_2, ..., X_n, ... is an infinite sequence of random variables, we can consider the infinite series $\sum_{i=1}^\infty X_i$ in the mean-square sense: by definition, the series converges in mean (square) if there exists the limit of sums from 1 to n, called the sum of the series:

$$\sum_{i=1}^\infty X_i = \operatorname*{l.i.m.}_{n\to\infty} \sum_{i=1}^n X_i. \tag{7-8.8}$$

Microtheorem 7-8.1. *If $\sum_{i=1}^\infty \|X_i\|_2 < \infty$, then the series $\sum_{i=1}^\infty X_i$ converges in mean squares.*

Proof. We have to prove that the sequence of the sums $Y_n = \sum_{i=1}^n X_i$ is a Cauchy sequence:

$$\operatorname*{l.i.m.}_{\substack{n\to\infty \\ m\to\infty}} (Y_n - Y_m) = 0; \tag{7-8.9}$$

in other words, that

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \|Y_n - Y_m\|_2 = 0. \tag{7-8.10}$$

It is enough to consider $m \geq n$ here; and for $m \geq n$

$$\|Y_n - Y_m\|_2 = \Big\| \sum_{i=n+1}^{m} X_i \Big\|_2 \leq \sum_{i=n+1}^{m} \|X_i\|_2. \tag{7-8.11}$$

We have to prove that for every positive ε there exists a natural N such that for every pair n, $m \geq N$

$$\|Y_n - Y_m\|_2 < \varepsilon. \tag{7-8.12}$$

Take N so that the "tail" of the series $\sum_{i=1}^{\infty} \|X_i\|_2$

$$\sum_{i=N+1}^{\infty} \|X_i\|_2 = \sum_{i=1}^{\infty} \|X_i\|_2 - \sum_{i=1}^{N} \|X_i\|_2 < \varepsilon. \tag{7-8.13}$$

Then for $m \geq n \geq N$

$$\|Y_n - Y_m\|_2 \leq \sum_{i=n+1}^{m} \|X_i\|_2 \leq \sum_{i=N+1}^{\infty} \|X_i\|_2 < \varepsilon. \tag{7-8.14}$$

We can reformulate Microtheorem 7-8.1 in terms of sequences instead of series:
Let Y_0, Y_1, Y_2, ..., Y_n, ... *be a sequence of random variables. If*

$$\sum_{i=1}^{\infty} \|Y_i - Y_{i-1}\|_2 < \infty, \tag{7-8.15}$$

then the mean-square limit $l.i.m._{n\to\infty} Y_n$ *exists.*

We just apply Microtheorem 7-8.1 to the sequence $X_i = Y_i - Y_{i-1}$.

It is easy to prove that

$$\Big\|\sum_{i=1}^{\infty} X_i \Big\|_2 \leq \sum_{i=1}^{\infty} \|X_i\|_2. \tag{7-8.16}$$

Another generalization of inequality (7-8.7): If X_t, $t \in [a, b]$, is a random function, then

$$\Big\| \int_a^b X_t\, dt \Big\|_2 \leq \int_a^b \|X_t\|_2\; dt. \tag{7-8.17}$$

This is proved using the fact that the integral is the limit of Riemann sums, to which we can apply formula (7-8.7).

Now we go to the stochastic equations.

But first let me remind you the situation in the ordinary differential equations.

One of approaches to differential equations is to replace the initial-value problem

$$\frac{dX_t}{dt} = b(t, X_t), \qquad X_{t_0} = x_0 \tag{7-8.18}$$

with the integral equation equivalent to it:

$$X_t = x_0 + \int_{t_0}^{t} b(s, X_s)\ ds. \tag{7-8.19}$$

Suppose the right-hand side function $b(t,x)$ is continuous in (t,x) and satisfies a Lipschitz condition in its second argument:

$$|b(t,x) - b(t,y)| \le C \cdot |x - y|, \tag{7-8.20}$$

where C is a constant.

The approach that I am about to present here can also be applied if the Lipschitz constant C si not a constant, but depends on t:

$$|b(t,x) - b(t,y)| \le C(t) \cdot |x - y|, \tag{7-8.21}$$

where $C(t)$ is a continuous function – because in every finite interval $C(t) \le \text{const}$.

Then the existence can be proved by using *successive approximations*:

$$X_t^{(0)} \equiv x_0, \tag{7-8.22}$$

and for $n \ge 1$

$$X_t^{(n)} = x_0 + \int_{t_0}^{t} b(s, X_s^{(n-1)})\ ds. \tag{7-8.23}$$

We have for $|t - t_0| \le T$:

$$|X_t^{(1)} - X_t^{(0)}| = \left|\int_{t_0}^{t} b(s, x_0)\ ds\right| \le K \cdot |t - t_0|; \tag{7-8.24}$$

for $n > 1$ we have (for a little shorter notations, we consider only $t \ge t_0$):

$$|X_t^{(n)} - X_t^{(n-1)}| = \left|\int_{t_0}^{t} \big(b(s, X_s^{(n-1)}) - b(s, X_s^{(n-2)})\big)\ ds\right| \le \int_{t_0}^{t} C \cdot |X_s^{(n-1)} - X_s^{(n-2)}|\ ds, \tag{7-8.25}$$

and by induction we prove that

$$|X_t^{(n)} - X_t^{(n-1)}| \le K \cdot C^{n-1} \cdot \frac{|t - t_0|^n}{n!}. \tag{7-8.26}$$

Since the series

$$\sum_{n=1}^{\infty} K \cdot C^{n-1} \cdot \frac{|t - t_0|^n}{n!} = \frac{K}{C} \cdot e^{C|t - t_0|} \tag{7-8.27}$$

converges, we see that the series

$$x_0 + (X_t^{(1)} - X_t^{(0)}) + (X_t^{(2)} - X_t^{(1)}) + ... + (X_t^{(n)} - X_t^{(n-1)}) + ... \tag{7-8.28}$$

converges, uniformly in t changing in every finite interval, that is, there exists the limit

$$X_t = \lim_{n\to\infty} X_t^{(n)}. \tag{7-8.29}$$

Limit passage in (7-8.23) yields (7-8.19).

As for proving the uniqueness of the solution, we are able to get at the same price also its continuous dependence on the initial condition x_0: if X_t is a solution of the initial-value problem (7-8.18), or, which is the same, of the integral equation (7-8.19), and Y_t is the solution of

$$\frac{dY_t}{dt} = b(t, Y_t), \qquad Y_{t_0} = y_0, \tag{7-8.30}$$

then we have:

$$|X_t - Y_t| \le |x_0 - y_0| \cdot e^{C|t-t_0|}. \tag{7-8.31}$$

This is obtained by subtraction (7-8.19) and the corresponding integral equation for Y_t (again I'll write this only for $t \ge t_0$ even if the same is true for $t < t_0$):

$$X_t - Y_t = (x_0 - y_0) + \int_{t_0}^{t} \big(b(s, X_s) - b(s, Y_s)\big)\, ds, \tag{7-8.32}$$

$$|X_t - Y_t| \le |x_0 - y_0| + \int_{t_0}^{t} C \cdot |X_s - Y_s|\, ds \tag{7-8.33}$$

This is an integral *inequality*; it turns out that its every solution is less or equal that the solution of the corresponding integral *equation*:

$$|X_t - Y_t| \le U_t, \tag{7-8.34}$$

where

$$U_t = |x_0 - y_0| + \int_{t_0}^{t} C \cdot U_s\, ds. \tag{7-8.35}$$

The equation (7-8.35) is equivalent to the initial-value problem

$$\frac{dU_t}{dt} = C \cdot U_t, \qquad U_{t_0} = |x_0 - y_0|, \tag{7-8.36}$$

so

$$U_t = |x_0 - y_0| \cdot e^{C(t-t_0)} \tag{7-8.37}$$

(I'll remind you that we consider now only the case of $t \ge t_0$).

To prove (7-8.34) it is enough to prove that for every $\varepsilon > 0$, $t \ge t_0$

$$|X_t - Y_t| < (|x_0 - y_0| + \varepsilon) \cdot e^{C(t-t_0)}. \tag{7-8.38}$$

At $t = t_0$ this is true. Suppose at some $t_1 > t_0$ it is not. The left-hand side of (7-8.38) and the right-hand side are continuous functions, so there is the smallest t_* such that for $t = t_*$ (7-8.38) does not hold; at this point the left-hand side is *equal* to the right-hand side. We have:

$$\begin{aligned} 0 &= |X_{t_*} - Y_{t_*}| - (|x_0 - y_0| + \varepsilon) \cdot e^{C(t_* - t_0)} \\ &\leq |x_0 - y_0| + \int_{t_0}^{t_*} C \cdot |X_s - Y_s| \, ds \\ &\qquad - (|x_0 - y_0| + \varepsilon) - \int_{t_0}^{t_*} C \cdot (|x_0 - y_0| + \varepsilon) \cdot e^{C(s-t_0)} \, ds \\ &< -\varepsilon < 0, \end{aligned} \tag{7-8.39}$$

because $|X_s - Y_s| < C \cdot e^{C(s-t_0)}$ for all $s \in [t_0, t^*)$. The contradiction obtained proves that it is impossible that inequality (7-8.34) shouldn't hold.

The uniqueness is a particular case with $|x_0 - y_0| = 0$.

We know that there are other theorems of existence (and uniqueness) for ordinary differential equations and that the Lipschitz condition (7-8.20) is not *necessary* for the existence and uniqueness to take place.

Now we go to stochastic equations. Let $b(t, x)$, $c(t, x)$, $t \geq t_0$, be two continuous functions. The stochastic differential equation

$$dX_t = b(t, X_t) \, dt + c(t, X_t) \, dW_t, \qquad t \geq t_0, \tag{7-8.40}$$

with the initial condition

$$X_{t_0} = x_0 \tag{7-8.41}$$

is, by definition, the same as the stochastic integral equation

$$X_t = x_0 + \int_{t_0}^{t} b(s, X_s) \, ds + \int_{t_0}^{t} c(s, X_s) \, dW_s \tag{7-8.42}$$

(and the stochastic process X_t is supposed to be determined by the past of the Wiener process and to have, almost surely, continuous trajectories $X_t(\omega)$).

Let us prove that if the continuous functions $b(t, x)$ and $c(t, x)$ satisfy a Lipschitz condition

$$|b(t, x) - b(t, y)| \leq C \cdot |x - y|, \qquad |c(t, x) - c(t, y)| \leq C \cdot |x - y|, \tag{7-8.43}$$

then a solution X_t, $t \geq t_0$, of (7-8.42) exists.

Note that, in contrast with the case of ordinary differential equations, we cannot solve the equation "backwards", from $t = t_0$ to the left: it turns out to be impossible to define the stochastic integral in the reversed time direction so that it has the same properties.

We are going to do it by successive approximations:

$$X_t^{(0)} \equiv x_0, \tag{7-8.44}$$

$$X_t^{(n)} = x_0 + \int_{t_0}^{t} b(s, X_s^{(n-1)})\ ds + \int_{t_0}^{t} c(s, X_s^{(n-1)})\ dW_s, \quad n = 1, 2, 3, \ldots . \tag{7-8.45}$$

But, while in the case of ordinary differential (integral) equations everything was simple: every next integral was taken of a continuous function – in the case of a stochastic equation there are many conditions that a random integrand has to satisfy for the stochastic integral to make sense; and some different (but simpler) conditions are needed in order for the Riemann integral $\int_{t_0}^{t} b(s, X_s^{(n-1)})\ ds$ to make sense.

We need to prove that for every $n = 0, 1, 2, \ldots$ the random function $X_t^{(n)}$

- is determined by the past (of the Wiener process);
- $E\big((X_t^{(n)})^2\big) < \infty$;
- is mean-square continuous in t;
- has almost surely continuous trajectories $X_t^{(n)}(\omega)$.

This is proved by induction.

And it's enough to prove it only in every finite interval from t_0 to $t_0 + T$. For $t \in [t_0, t_0 + T]$ we have:

$$|b(t,0)|, |c(t,0)| \le K, \qquad |b(t,x)|, |c(t,x)| \le K + C \cdot (t - t_0), \tag{7-8.46}$$

where K is some constant.

The basis for the induction is $n = 0$: the random function $X_t^{(0)} \equiv x_0$ is clearly determined by the past (to know the value of $X_t^{(0)}$ it's not only enough to know the values of W_s for $t_0 \le s \le t$, but even enough to know nothing at all), and all remaining properties are satisfied in a trivial way.

Now suppose that the above conditions are satisfied with $n - 1$ in lieu of n; we have to check that they are satisfied also with n.

That the random function (7-8.45) is determined by the values of the Wiener process up to time t is clear: the integrands in both integrals are determined by W_s, $t_0 \le s \le t$. About finiteness of $E\big((X_t^{(n)})^2\big)$: by (7-8.7), (7-8.17),

$$\|X_t^{(n)}\|_2 \le |x_0| + \int_{t_0}^{t} \|b(s, X_s^{(n-1)})\|_2\ ds + \sqrt{\Big(\big(\int_{t_0}^{t} c(s, X_s^{(n-1)})\ dW_s\big)^2\Big)}. \tag{7-8.47}$$

The expectation under the square root is equal to

$$\begin{aligned} \int_{t_0}^{t} E\Big(\big(c(s, X_s^{(n-1)})\big)^2\Big)\ ds &\le \int_{t_0}^{t} E\big((K + C \cdot |X_s^{(n-1)}|)^2\big)\ ds \\ &\le \int_{t_0}^{t} \big[2K^2 + 2C^2 E\big((X_s^{(n-1)})^2\big)\big]\ ds < \infty, \end{aligned} \tag{7-8.48}$$

because the quadratic expectation of a mean-square continuous random function $X_s^{(n-1)}$ is bounded in the finite interval $[t_0, t_0 + T]$. For the non-stochastic integral we also have:

$$\int_{t_0}^{t} \|b(s, X_s^{(n-1)})\|_2\ ds \le \int_{t_0}^{t} [K + C \cdot \|X_s^{(n-1)}\|_2]\ ds < \infty. \tag{7-8.49}$$

Mean-square continuity: for $t_0 \le t \le t' \le t_0 + T$

$$\begin{aligned}
\|X_{t'}^{(n)} - X_t^{(n)}\|_2 &= \Big\| \int_t^{t'} b(s, X_s^{(n-1)})\, ds + \int_t^{t'} c(s, X_s^{(n-1)})\, dW_s \Big\|_2 \\
&\le \int_t^{t'} \|b(s, X_s^{(n-1)})\|_2\, ds + \sqrt{E\Big(\big(\int_t^{t'} c(s, X_s^{(n-1)})\, dW_s\big)^2\Big)} \\
&\le \int_t^{t'} [K + C \cdot \|X_s^{(n-1)}\|_2]\, ds \\
&\qquad + \sqrt{\int_t^{t'} \big[2K^2 + 2C^2 E\big((X_s^{(n-1)})^2\big)\big]\, ds} \\
&\to 0 \qquad (t' - t \to 0).
\end{aligned} \tag{7-8.50}$$

As for almost-sure continuity of the trajectories $X_t^{(n)}$, we leave it now unproved, and perhaps we'll return to it in the future (after we introduce *martingales*, which were announced in the short description of the course). Anyway, we have a big thing in this field left without a proof: the continuity of the trajectories of a Wiener process, so one little proof more or one less makes not much difference.

Now we go to the estimates of differences of successive approximations, similar to (7-8.26).

We are going to prove that for every $n = 1, 2, 3, \ldots$ and every $t \in [t_0, t_0 + T]$

$$\|X_t^{(n)} - X_t^{(n-1)}\|_2 \le C_1 \cdot C_2^{n-1} \cdot \frac{(t - t_0)^{n/2}}{\sqrt{n!}}, \tag{7-8.51}$$

where C_1, C_2 are some constants; or, which is the same,

$$E\big((X_t^{(n)} - X_t^{(n-1)})^2\big) \le C_1^2 \cdot C_2^{2(n-1)} \cdot \frac{(t - t_0)^n}{n!}. \tag{7-8.52}$$

We'll prove (7-8.51), (7-8.52) by induction. The basis is $n = 1$:

$$\begin{aligned}
\|X_t^{(1)} - X_t^{(0)}\|_2 &= \Big\| \int_{t_0}^t b(s, x_0)\, ds + \int_{t_0}^t c(s, x_0)\, dW_s \Big|_2 \\
&\le \int_{t_0}^t |b(s, x_0)|\, ds + \sqrt{\int_{t_0}^t c(s, x_0)^2\, ds} \\
&\le (K + |x_0|) \cdot (t - t_0) + \sqrt{(K + |x_0|)^2 \cdot (t - t_0)} \\
&\le C_1 \cdot (t - t_0)^{1/2},
\end{aligned} \tag{7-8.53}$$

where

$$C_1 = (K + |x_0|) \cdot (1 + \sqrt{T}); \tag{7-8.54}$$

and (7-8.51), (7-8.52) is satisfied for $n = 1$.

Suppose these inequalities hold for $n-1$ instead of n; let us estimate $\|X_t^{(n)} - X_t^{(n-1)}\|_2$. We have:

$$
\begin{aligned}
\|X_t^{(n)} - X_t^{(n-1)}\|_2 &= \Big\| \int_{t_0}^t [b(s, X_s^{(n-1)}) - b(s, X_s^{(n-2)})]\ ds \\
&\qquad + \int_{t_0}^t [c(s, X_s^{(n-1)}) - c(s, X_s^{(n-2)})]\ dW_s \Big\|_2 \\
&\le \int_{t_0}^t \|b(s, X_s^{(n-1)}) - b(s, X_s^{(n-2)})\|_2\ ds \\
&\qquad + \sqrt{\int_{t_0}^t E\big(\big(c(s, X_s^{(n-1)}) - c(s, X_s^{(n-2)})\big)^2\big)\ ds} \\
&\le C \cdot \int_{t_0}^t \|X_s^{(n-1)} - X_s^{(n-2)}\|_2\ ds \\
&\qquad + \sqrt{C^2 \cdot \int_{t_0}^t E\big(\big(X_s^{(n-1)} - X_s^{(n-2)}\big)^2\big) ds} \\
&\le C \cdot C_1 \cdot C_2^{n-2} \cdot \Big(\frac{(t-t_0)^{n/2+1}}{\sqrt{(n-1)!} \cdot (n/2+1)} + \sqrt{\frac{(t-t_0)^n}{n!}} \Big).
\end{aligned}
\tag{7-8.55}
$$

Since $(n-1)! \cdot (n/2+1)^2 \ge n!/4$, $(t-t_0)^{n/2+1} \le \sqrt{T} \cdot (t-t_0)^{n/2}$, we have:

$$
\|X_t^{(n)} - X_t^{(n-1)}\|_2 \le C \cdot (1 + 2\sqrt{T}) \cdot C_1 \cdot C_2^{n-2} \cdot \frac{(t-t_0)^{n/2}}{\sqrt{n!}}. \tag{7-8.56}
$$

If we choose

$$
C_2 = C \cdot (1 + 2\sqrt{T}), \tag{7-8.57}
$$

we have the inequality (7-8.52) proved.

Now, the series

$$
\sum_{n=1}^{\infty} C_1 \cdot C_2^{n-1} \cdot \frac{(t-t_0)^{n/2}}{\sqrt{n!}} \tag{7-8.58}
$$

converges (uniformly in $t \in [t_0, t_0 + T]$), even if its sum is not expressed in terms of an exponential function, as in (7-8.27); therefore there exists the mean-square limit

$$
X_t = \operatorname*{l.i.m.}_{n \to \infty} X_t^{(n)}, \tag{7-8.59}
$$

determined by the past of the Wiener process, and having a finite quadratic expectation. This convergence is uniform in $t \in [t_0, t_0 + T]$:

$$
\max_{t_0 \le t \le t_0 + T} \|X_t^{(n)} - X_t\|_2 \le \sum_{i=n+1}^{\infty} C_1 \cdot C_2^{i-1} \cdot \frac{(t-t_0)^{i/2}}{\sqrt{i!}} \to 0 \qquad (n \to \infty); \tag{7-8.60}
$$

it follows from this that the limiting random function is mean-square continuous, and the stochastic integral $\int_{t_0}^{t} c(s, X_s)\ dW_s$ makes sense.

To handle the first integral in (7-8.42), which we understand as a Riemann integral $\int_{t_0}^{t} b\big(s, X_s(\omega)\big)\ ds$, taken separately for each (rather, *almost* each) $\omega \in \Omega$, we need to have our trajectories $X_t(\omega)$ continuous for all t for almost all ω (almost surely); and for this, almost-sure uniform convergence of the trajectories $X_t^{(n)}$ is sufficient. I'll leave this without proof, together with other statements having to do with trajectories' continuity (in particular, Theorem 6.1).

So as $n \to \infty$, the random functions $X_t^{(n-1)}$ almost surely converge to X_t uniformly for $t \in [t_0,\, t_0 + T]$; so the Riemann integral $\int_{t_0}^{t} b(s, X_s^{(n-1)})\ ds$ converges almost surely to $\int_{t_0}^{t} b(s, X_s)\ ds$. The stochastic integral $\int_{t_0}^{t} c(s, X_s^{(n-1)})\ dW_s$ converges in the mean squares to $\int_{t_0}^{t} c(s, X_s)\ dW_s$. From both of these kinds of convergence follows convergence in probability; so, as $n \to \infty$, the right-hand side of (7-8.45) converges in probability to the right-hand side of (7-8.42). And the left-hand side of (7-8.45) converges to that of (7-8.42); so by the almost-uniqueness of the limit in probability (Microtheorem 1.2) we have that the equation (7-8.42) is satisfied almost surely.

The existence of a solution of a stochastic equation, under the Lipschitz condition on the coefficients $b(t, x)$, $c(t, x)$ is proved.

Now to the uniqueness.

Suppose that X_t, $t \geq t_0$, is a mean-square continuous solution of (7-8.42) with $E(X_t^2) < \infty$, and Y_t is a mean-square solution of the same equation with a different initial condition:

$$Y_t = y_0 + \int_{t_0}^{t} b(s, Y_s)\ ds + \int_{t_0}^{t} c(s, Y_s)\ dW_s, \qquad \text{(7-8.61)}$$

$E(Y_t^2) < \infty$. We are going to prove that for $t_0 \leq t \leq t_0 + T$

$$\|X_t - Y_t\|_2 \leq |x_0 - y_0| \cdot F(t), \qquad \text{(7-8.62)}$$

where $F(t)$ is some continuous function; which means that

$$E\big((X_t - Y_t)^2\big) \leq (x_0 - y_0)^2 \cdot F(t)^2. \qquad \text{(7-8.63)}$$

We have:

$$
\begin{aligned}
\|X_t - Y_t\|_2 = |x_0 - y_0| + \Big\| \int_{t_0}^t [b(s, X_s) - b(s, Y_s)]\ ds + \int_{t_0}^t [c(s, X_s) - c(s, Y_s)]\ dW_s \Big\|_2 \\
\le |x_0 - y_0| + \int_{t_0}^t \|b(s, X_s) - b(s, Y_s)\|_2\ ds + \sqrt{\int_{t_0}^t E\big(\big(c(s, X_s) - c(s, Y_s)\big)^2\big)\ ds} \\
\le |x_0 - y_0| + C \cdot \Big[\int_{t_0}^t \|X_s - Y_s\|_2\ ds + \sqrt{\int_{t_0}^t E\big((X_s - Y_s)^2\big)\ ds}\Big].
\end{aligned}
\tag{7-8.64}
$$

Let us estimate the first integral through the second:

$$
\begin{aligned}
\int_{t_0}^t \|X_s - Y_s\|_2\ ds &= \sqrt{\int_{t_0}^t \int_{t_0}^t \|X_s - Y_s\|_2 \cdot \|X_{s'} - Y_{s'}\|_2\ ds\, ds'} \\
&\le \sqrt{\int_{t_0}^t \int_{t_0}^t \frac{1}{2}[\|X_s - Y_s\|_2^2 + \|X_{s'} - Y_{s'}\|_2^2]\ ds\, ds'} \\
&= \sqrt{(t - t_0) \cdot \int_{t_0}^t E\big((X_s - Y_s)^2\big)\ ds} \\
&\le \sqrt{T} \cdot \sqrt{\int_{t_0}^t E\big((X_s - Y_s)^2\big)\ ds},
\end{aligned}
\tag{7-8.65}
$$

$$
E\big((X_t - Y_t)^2\big) \le 2(x_0 - y_0)^2 + 2C^2 \cdot (1 + \sqrt{T})^2 \cdot \int_{t_0}^t E\big((X_s - Y_s)^2\big)\ ds. \tag{7-8.66}
$$

The function $E\big((X_t - Y_t)^2\big)$ satisfies the integral inequality of the same kind as (7-8.33), with $2(x_0 - y_0)^2$ instead of $|x_0 - y_0|$ and $2C^2 \cdot (1 + \sqrt{T})^2$ instead of C; so, similarly to (7-8.34), (7-8.37), we have:

$$
E\big((X_t - Y_t)^2\big) \le 2(x_0 - y_0)^2 \cdot e^{C^2 (1+\sqrt{T})^2 (t-t_0)}. \tag{7-8.67}
$$

This proves the uniqueness of a solution with finite $E(X_t^2)$; and at the same time, mean-square continuous dependence of the solution on the initial condition.

The same results of existence and uniqueness are proved in the same way for *systems* of stochastic equations

$$
dX_t^i = b_i\big(t, \boldsymbol{X}_t\big)\, dt + \sum_{k=1}^n c_{ik}\big(t, \boldsymbol{X}_t\big)\, dW_t^k, \qquad i = 1, ..., r, \tag{7-8.68}
$$

with initial conditions

$$
X_{t_0}^i = x_0^i, \tag{7-8.69}
$$

or, in the form of integral equations,

$$X^i(t) = x_0 + \int_{t_0}^{t} b_i\big(s, \boldsymbol{X}_s\big)\, ds + \sum_{k=1}^{n} \int_{t_0}^{t} c_{ik}\big(s, \boldsymbol{X}_s\big)\, dW_s^k; \tag{7-8.70}$$

or, in the vector-matrix form,

$$d\boldsymbol{X}_t = \boldsymbol{b}(t, \boldsymbol{X}_t)\, dt + C(t, \boldsymbol{X}_t)\, dW_t, \qquad \boldsymbol{X}_{t_0} = \boldsymbol{x}_0, \tag{7-8.71}$$

$$\boldsymbol{X}_t = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{b}(s, \boldsymbol{X}_s)\ ds + \int_{t_0}^{t} C(s, \boldsymbol{X}_s)\ d\boldsymbol{W}_s, \tag{7-8.72}$$

if the coefficients, the column vector-valued function $\boldsymbol{b}(s, \boldsymbol{x})$, $t \geq t_0$, $\boldsymbol{x} \in \mathbb{R}^r$, with components $b_1(t, \boldsymbol{x})$, ..., $b_r(t, \boldsymbol{x})$ and the $(r \times n)$-matrix valued $C(t, \boldsymbol{x}) = (c_{ik})_{i=1,\,\dots,\,r,\; k=1,\,\dots,\,n}$ are continuous in $(t, \boldsymbol{x})$ and satisfy Lipschitz conditions in $\boldsymbol{x}$:

$$|b_i(t, \boldsymbol{x})) - b_i(t, \boldsymbol{y})| \leq C \cdot |\boldsymbol{x} - \boldsymbol{y}|, \qquad |c_{ik}(t, \boldsymbol{x})) - c_{ik}(t, \boldsymbol{y})| \leq C \cdot |\boldsymbol{x} - \boldsymbol{y}| \tag{7-8.73}$$

But, of course, writing all the formulas would take too much space in this case.