Existence of conditional expectations.

I think, I was wrong underlining again and again the the question of existence of conditional expectations and probabilities, being important in the theory, is not important in applications, where their existence is postulated, together with their concrete form, and postponing again and again facing the question of existence. *Now* seems to be the wrong time; and any time in the future too. So I am making of it a separate note.

Theorem Ex.1. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space, and* $\mathcal{A}$ *a sub-*σ*-algebra of* $\mathcal{F}$: $\mathcal{A} \subseteq \mathcal{F}$. *Let* ξ *be a random variable having a finite expectation*: $E|\xi| < \infty$.

Then the conditional expectation $E(\xi \| \mathcal{A})$ *exists.*

The **proof** will be based on a theorem of the theory of measure and integration – of which I will give the formulation here, without proof:

Theorem Ex.2 (Radon – Nikodym Theorem). *Let* $(X, \mathcal{X})$ *be a measurable space*; m *a* σ*-finite measure on it. Let* n *be a finite* signed measure (a countably additive set function) *on the* σ*-algebra* $\mathcal{X}$, *i. e. a function* $n\colon \mathcal{X} \mapsto \mathbb{R}^1$ *such that* $n\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} n(A_i)$ *for every sequence of disjoint sets* $A_i \in \mathcal{X}$.

Then there exists an $\mathcal{X}$*-measurable function* $f(x)$, $x \in X$, (called the *density* of n with respect to m) *such that* $n(A) = \int_A f(x)\ m(dx)$ *for every set* $A \in \mathcal{X}$.

Proof of Theorem Ex.1. Define the set function

$$n(A) = \int_A \xi(\omega)\ P(d\omega) = E[I_A \cdot \xi]. \tag{Ex.1}$$

This function is defined and real-valued (i. e., finite) for all sets $A \in \mathcal{F}$; and it is a signed measure. Indeed, for disjoint sets A_i we have:

$$I_{\bigcup_{i=1}^{\infty} A_i} \cdot \xi = \sum_{i=1}^{\infty} I_{A_i} \cdot \xi; \tag{Ex.2}$$

all partial sums $\sum_{i=1}^{n} I_{A_i} \cdot \xi$ are dominated in absolute value by the integrable random variable $|\xi|$, and the dominated-convergence theorem yields what we need.

Now let us consider this set function and the probability measure P *only on the* σ*-algebra* $\mathcal{A}$. Applying Theorem Ex.2 with P instead of m, Ω instead of X, ω instead of x, and $\mathcal{A}$ as $\mathcal{X}$, we get that there exists an $\mathcal{A}$-measurable function $\varphi(\omega)$ (an $\mathcal{A}$-measurable random variable) such that $n(A) = \int_A \varphi(\omega)\ P(d\omega)$ for all $A \in \mathcal{A}$. But this means exactly that

$$\int_A \xi\ dP = \int_A \varphi\ dP \tag{Ex.3}$$

for every $A \in \mathcal{A}$, which, together with the $\mathcal{A}$-measurability of φ, is what is required in the definition of the conditional expectation.

The proof is so simple (after it has been invented) that it's difficult to understand and to accept (one is looking for a bad trick here).