MATH 657 STOCHASTIC DIFFERENTIAL EQUATIONS SPRING 2007

Problems.

1 Let X_1, Y_1, X_2, Y_2 be independent random variables having each the normal distribution with parameters $(0, 1)$. For every real t, let

$$Z_t = X_1 \cos t + Y_1 \sin t + X_2 \cos 2t + Y_2 \sin 2t. \qquad (1_1)$$

Find the joint distribution $\mu_{Z_{t_1}, Z_{t_2}}$ of the values Z_{t_1}, Z_{t_2} of this process for $t_1 = \pi/6$, $t_2 = \pi/2$.

Hint: This distribution is Gaussian.

2 Suppose the random variables X_1, X_3 have a discrete joint distribution given by

$$p_{13}(x_1, x_3) = P\{X_1 = x_1, X_3 = x_3\} = \pi_1(x_1) \cdot \pi_2(x_3 - x_1),$$

where $\pi_a(x)$ is the probability mass function of the Poisson distribution with parameter a:

$$\pi_a(x) = \begin{cases} \dfrac{a^x e^{-a}}{x!} & \text{if } x \text{ is a nonnegative integer,} \\ 0 & \text{otherwise;} \end{cases}$$

and X_0 is the random variable that is identically equal to 0.

Show that the random variables $X_1 - X_0$, $X_3 - X_1$ are independent, and find their one-dimensional distributions.

3 Let a system of finite-dimensional distributions $\mu_{t_1, t_2, \dots, t_n}(C)$, $0 \le t_1 < t_2 < \dots < t_n$, be defined by

$$\mu_{t_1, t_2, \dots, t_n}(C) = \sum_{(x_1, x_2, \dots, x_n) \in C} \pi_{\lambda(t_1 - t_0)}(x_1 - x_0) \cdot \pi_{\lambda(t_2 - t_1)}(x_2 - x_1) \cdot \dots \cdot \pi_{\lambda(t_n - t_{n-1})}(x_n - x_{n-1}), \qquad (3_1)$$

where $t_0 = 0$, $x_0 = 0$, and λ is a positive number.

Does this system satisfy the consistency conditions? In other words, does there exist a stochastic process with its finite-dimensional distributions given by (3_1)?

If yes, does there exist such a process *with continuous trajectories*?

4 Suppose the random variables X_1, X_3 have a continuous joint distribution with density given by

$$p_{13}(x_1, x_3) = q_1(x_1) \cdot q_2(x_3 - x_1),$$

where $q_a(x)$ is the density of the uniform distribution on the interval $[0, a]$:

$$q_a(x) = \begin{cases} 1/a & \text{for } 0 \le x \le a, \\ 0 & \text{for } x \notin [0, a]; \end{cases}$$

and X_0 is the random variable that is identically equal to 0.

Show that the random variables $X_1 - X_0$, $X_3 - X_1$ are independent, and find their one-dimensional distributions.

5 Let a system of finite-dimensional distributions $\mu_{t_1,\,t_2,\,...,\,t_n}(C)$, $0 \le t_1 < t_2 < ... < t_n$, be defined by

$$\mu_{t_1,\,t_2,\,...,\,t_n}(C) = \int_C \cdots \int q_{t_1-t_0}(x_1 - x_0) \cdot q_{t_2-t_1}(x_2 - x_1) \cdot ... \cdot q_{t_n-t_{n-1}}(x_n - x_{n-1})\; dx_1 \,...\, dx_n, \tag{5_1}$$

where $t_0 = 0$, $x_0 = 0$.

Does this system satisfy the consistency conditions? In other words, does there exist a stochastic process with its finite-dimensional distributions given by (5_1)?

If yes, does there exist such a process *with continuous trajectories*?

6 Let W_t^1, W_t^2, $t \ge t_0$, be two independent Wiener processes (as, e. g., two different coordinates of a multidimensional Wiener process. Their being independent means that for every finite collection of time moments $t_0 < t_1 < t_2 < ... < t_n$ the random vectors $(W_{t_0}^1, W_{t_1}^1, W_{t_2}^1, ..., W_{t_n}^1)$ and $(W_{t_0}^2, W_{t_1}^2, W_{t_2}^2, ..., W_{t_n}^2)$ are independent). For a partition $\mathfrak{T}$ of the interval from $a = t_0$ to b with partition points $t_0 = a < t_1 < t_2 < ... < t_n = b$, take

$$\Sigma_{\mathfrak{T}}^{1,\,2} = \sum_{i=1}^{n} (W_{t_i}^1 - W_{t_{i-1}}^1) \cdot (W_{t_i}^2 - W_{t_{i-1}}^2).$$

Prove that there exists a mean-square limit

$$\underset{\max_{1 \le t \le n}(t_i - t_{i-1}) \to 0}{\text{l.i.m.}} \Sigma_{\mathfrak{T}}^{1,\,2}.$$

Is this limit a constant or a non-constant random variable?

If a constant, what is it equal to? If non-constant, what can you say about its distribution?

7 Let T be a positive random variable having the exponential distribution with parameter 1: that is, with probability density

$$p_T(t) = \begin{cases} e^{-t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Let the random function $f(t, \omega)$, $t \ge 0$, be defined by

$$f(t, \omega) = \begin{cases} 1, & t < T(\omega), \\ 0, & t \ge T(\omega). \end{cases}$$

The realizations (sample functions) of this random function are all discontinuous (namely, at the point $t = T(\omega)$).

Is the random function $f(t, \omega)$ mean-square continuous in the interval $[0, \infty)$?

Is it continuous in the sense of convergence in probability on this interval?

For many of the following problems you *need* to (actively) know what a multidimensional normal (Gaussian) distribution is.

8 Let $\boldsymbol{W}_t = (W_t^1, ..., W_t^r)$ be an r-dimensional Wiener process. Prove that if O is an $(r \times r)$ orthogonal matrix, then $\tilde{\boldsymbol{W}}_t = O\boldsymbol{W}_t$ is another Wiener process.

(This is the multidimensional analog of the one-dimensional fact that $-W_t$ is also a Wiener process.)

9 Prove that for

$$V_t = v_0 e^{-a(t-t_0)} + \sigma \int_{t_0}^{t} e^{-a(t-s)}\, dW_s, \qquad t \geq t_0, \tag{9_1}$$

we have:

$$dV_t = -aV_t\, dt + \sigma\, dW_t.$$

10 Let $g(t)$ be a step non-random function:

$$f(t) = \sum_{i=1}^{n} c_i \cdot I_{(t_{i-1},\, t_i]}(t).$$

Prove that the stochastic integral $\int_a^b f(t)\ dW_t$ is a random variable that has a normal distribution. Find its parameters.

11 Let $g(t)$ be a non-step non-random function for which the stochastic integral $\int_a^b f(t)\ dW_t$ is a random variable that has a normal distribution. Find its parameters.

12 Let $X_t = X(t, \omega)$, $a \leq t \leq b$, be a mean-square continuous random function such that all joint distributions of X_{t_1}, X_{t_2}, ..., X_{t_n}, $t_i \in [a, b]$, are Gaussian. Prove that the Riemann integral $\int_a^b X_t\, dt$ is a random variable having a (one-dimensional) normal distribution.

13 Prove that for every $t \geq t_0$ the random variable V_t of Problem **9** has a normal distribution. Find its parameters.

13 a For the stochastic process V_t of Problem **9**, represent $X_t = x_0 + \int_{t_0}^{t} V_s\, ds$ in the form of a non-random summand plus a stochastic integral.

14 For the stochastic process V_t of Problem **9**, find the distribution of the random variable $X_t = x_0 + \int_{t_0}^{t} V_s\, ds$.

15 Prove that if we take in the Problems **9**, **14** the parameter $a = \sigma^2 \to \infty$, the distribution of the random variable $X_t - x_0$ converges weakly to some (one-dimensional) distribution. What is the limiting distribution?

16 The joint distribution of two random variables X, Y is uniform over the hexagon $\{(x,y)\colon |x| \le 1, |y| \le 1, |x+y| \le 1\}$. Find $E(X\|Y)$.

17 Prove that the stochastic process Y_t, $t \ge 0$, of Example 17.1 is a Markov process on the space $\mathrm{Sp} = \{0,1\}$. Find its transition (2×2) matrix P^{ts}, $t \le s$.

18 Let T be a positive random variable whose distribution is not exponential. Let the stochastic process Y_t, $t \ge 0$, be defined as

$$Y_t = \begin{cases} 1, & 0 \le t < T, \\ 0, & t \ge T. \end{cases}$$

Is Y_t a Markov process? If it is, what is its transition matrix?

19 Let P^{ts}, $t \le s$, be defined by

$$P^{ts} = \begin{pmatrix} \dfrac{2}{3} + \dfrac{e^{-3(s-t)}}{3} & \dfrac{1}{3} - \dfrac{e^{-3(s-t)}}{3} \\ \dfrac{2}{3} - \dfrac{2e^{-3(s-t)}}{3} & \dfrac{1}{3} + \dfrac{2e^{-3(s-t)}}{3} \end{pmatrix}.$$

Check that the entries of these matrices are nonnegative and add up to 1 in each row, and that Chapman – Kolmogorov equation is satisfied for this family of matrices (and so there exists a Markov process with this matrix as its transition matrix; what can you say about this process?).

20 For the stochastic process Y_t, $t \ge 0$, of Example 17.1, prove that T is a stopping time with respect to this process.

Check that for a non-random $t_* > 0$ also $\min(T, t_*)$ is a stopping time.

21 For the same process, define the following random function R_t, $t \ge 0$:

$$R_t = \begin{cases} Y_t + at, & t < T, \\ Y_T + aT, & t \ge T. \end{cases} = \begin{cases} 1 + at, & t < T, \\ aT, & t \ge T. \end{cases}$$

Prove that R_t, $t \ge 0$, is a martingale.

22 Check that

$$E(R_{\min(T,\,t_*)}) = E(R_{t_*})$$

(evaluate both these expectations and check that they are the same).

23 Let us define the random function L_t, $t \ge 0$, by $L_0 = R_0$ $(\equiv 1)$, and for $t > 0$

$$L_t = \lim_{s \to t-} R_s = \begin{cases} 1 + at, & t \le T, \\ aT, & t > T \end{cases}$$

(R_t was for "the *r*ight-continuous version", L_t for the *l*eft-continuous).

Check that for every t we have $P\{L_t = R_t\} = 1$. Check that L_t, $t \ge 0$, is a martingale.

24 Check that

$$E(L_{\min(T,\,t_*)}) \neq E(L_{t_*})$$

(evaluate both these expectations and check that they are not the same).

25 Let $b > 0$. For the diffusion process $X_t = W_t + bt$, similarly to how it was done for the Wiener process in Lecture note 31 (by limit passage), find, for $a < x < c$, the probabilities

$$P\{\tau^x_{(a,\,\infty)} < \infty\}, \qquad P\{\tau^x_{(-\infty,\,c)} < \infty\}.$$

26 Let $b \neq 0$. For the diffusion process $X_t = W_t + bt$, find the expected time spent in a finite interval before leaving it for the first time: $E(\tau^x_{(a,\,c)})$.

27 Let $b > 0$. For the diffusion process $X_t = W_t + bt$, similarly to how it was done for the Wiener process in Lecture note 31, find, for $c > x$, the expectation $E(\tau^x_{(-\infty,\,c)})$.

28 Check that the function $u(\boldsymbol{x}) = -\ln|\boldsymbol{x}|$, $\boldsymbol{x} \in \mathbb{R}^2$, satisfies the equation $\Delta u(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$.

29 Is the expectation $E\big(u(\boldsymbol{W}_t)\big)$ finite for the function u of the previous problem?

30 If it is, what is $\lim_{t\to\infty} E\big(u(\boldsymbol{W}_t)\big)$ equal to?

31 Is the random function $u(\boldsymbol{W}_t)$ of the previous problem a martingale?

32 Check that the function $u(\boldsymbol{x}) = |\boldsymbol{x}|^{2-r}$, $\boldsymbol{x} \in \mathbb{R}^r$, $r > 2$, satisfies the equation $\Delta u(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$.

33 For a bounded continuous function $\varphi(x)$, $-\infty < x < \infty$, let

$$u(x^1,\, x^2) = \int_{-\infty}^{\infty} \frac{\pi^{-1} \cdot x^2}{(x^2)^2 + (x^1 - y)^2} \cdot \varphi(y)\, dy, \qquad -\infty < x^1 < \infty,\ x^2 > 0 \tag{33$_1$}$$

(the superscript does not denote the power).

Check that this function satisfies the equation $\Delta u(x^1,\, x^2) = 0$.

34 Prove that for this function for every point $(x, 0)$ on the horizontal axis

$$\lim_{x^1 \to x,\, x^2 \to 0+} u(x^1,\, x^2) = \varphi(x).$$

In other words, the function (33_1) is the (bounded) solution of the Dirichlet problem $\Delta u(\boldsymbol{x}) = 0$ in the upper half-plane, $u(\boldsymbol{x}) = \varphi(\boldsymbol{x})$ on its boundary, where we put $\varphi(x^1,\, 0) = \varphi(x^1)$.

35 For the one-dimensional Wiener process, prove that the expectation $E\big((\tau^x_{(a,\,b)})^2\big) < \infty$.

36 For the one-dimensional Wiener process, is $E\big((\tau^x_{(a,\,b)})^3\big) < \infty$? Is $E\big((\tau^x_{(a,\,b)})^4\big) < \infty$?

37* (* meaning that the problem is non-obligatory) For the one-dimensional Wiener process, evaluate $E\big((\tau^x_{(a,\,b)})^2\big)$ for $a \le x \le b$.

38 For the one-dimensional Wiener process, is $E(e^{(\tau^x_{(a,\,b)})^2}) < \infty$?

39 For $\boldsymbol{W}^{\boldsymbol{x}}_t$ being the two-dimensional Wiener process, and the region G being the circle $\{\boldsymbol{x} = (x^1,\, x^2)\colon\, (x^1)^2 + (x^2)^2 < R^2\}$, find the expectation $E(\tau^{\boldsymbol{x}}_G)$ of the first exit time from G.

40 Same for G being the strip $\{(x^1,\, x^2)\colon\, |x^1| < a\},\ a > 0$.

41 Same for G being an ellipse $\big\{(x^1,\, x^2)\colon\, A \cdot (x^1)^2 + B \cdot (x^2)^2 < 1\big\},\ A,\, B > 0$.

42 For a one-dimensional controlled diffusion process $X^{x,\,u}_t$ with coefficients $\sigma(x,u) \equiv 1$ ($a(x,u) \equiv 1$), $b(x,u) = u$, where the control parameter u takes values in the set $U = [-1,1]$, and an interval $G = (-c,\, c)$, $c > 0$, find the control strategy $u(t;\, x_s,\, 0 \le s \le t)$ that maximizes

$$E\Big(\int_0^{\tau^x_{(-c,\,c)}} \big[1 - |u(t;\, X^{x,\,u}_s,\, 0 \le s \le t)|\big]\ dt\Big)$$

(that is, our gain per time unit is given by the function $g(x,u) = g(u) = 1 - |u|$). Note that the optimal strategy may have different character for different values of c. Find the maximum expected gain for the initial point $x = 0$.