1 Is the differential equation
$$\frac{dy}{dt} = t\,y^2$$
linear? Is it a linear homogeneous equation?

Is the function $y(t) \equiv 0$ a solution of this equation? Is $y(t) \equiv 1$ its solution?

Find the general solution of this equation.

Find its solution satisfying the initial condition $y(2) = 1$. On what interval (containing the point $t = 2$, of course) is this solution defined? What happens with $y(t)$ as t approaches the right-hand end of this interval? What happens with the solution as t approaches the left end of the interval in which it is defined?

Solution. Of course the equation is not linear, because it contains y^2 multiplied by a function of t. And of course, being nonlinear, it cannot be linear homogeneous.

Clearly the function $y(t) \equiv 0$ *is* a solution of our equation: $\frac{d0}{dt} \equiv 0 \equiv t \cdot 0^2$; and $y(t) \equiv 1$ is not: $\frac{d1}{dt} \equiv 0 \not\equiv t \cdot 1^2$.

As for the solutions of this – separable – equation, one solution is $y(t) \equiv 0$; for $y \neq 0$ we write:

$$\frac{dy}{y^2} = t\,dt, \qquad \int \frac{dy}{y^2} = \int t\,dt, \qquad -\frac{1}{y} = \frac{t^2}{2} + C, \qquad y(t) = -\frac{1}{t^2/2 + C},$$

where C is an arbitrary constant. So the general solution is:

$$y(t) \equiv 0 \qquad \text{or} \;\; y(t) = -\frac{1}{t^2/2 + C}.$$

Note that the second formula does not include the first one for any value of the constant C.

Now to the particular solution satisfying the initial condition $y(2) = 1$. This solution, of course, is not the identical zero, so we use the second formula in our general solution:

$$y(2) = -\frac{1}{2^2/2 + C} = 1,$$

so $2^2/2 + C = -1$, and $C = -3$. So the solution is given, in its interval I of existence, by the formula
$$y(t) = -\frac{1}{t^2/2 - 3}. \tag{1_1}$$

What is I, the interval of existence?

The right-hand side of (1_1) is defined and differentiable, with our equation satisfied, for $t \neq \pm\sqrt{6}$, that is, in three intervals: $(-\infty, -\sqrt{6})$, $(-\sqrt{6}, \sqrt{6})$, and $(\sqrt{6}, \infty)$. of these intervals, the middle one contains the point $t_0 = 2$; so the interval of existence is $(-\sqrt{6}, \sqrt{6})$.

As $t \to (\sqrt{6})^-$, we have $y(t) \to \infty$, and also $\lim_{t\to(-\sqrt{6})^+} y(t) = \infty$.

2 Consider the autonomous equation $\frac{dy}{dt} = y^3(\alpha - y)$.

(a) For each of the values of the parameter α, draw the picture of the phase line, showing the equilibrium points and the direction of the motion in the intervals between the equilibrium points. Which of the equilibrium points are asymptotically stable, and which unstable (the answer may be different for different α's)?

(b) For some value of α, draw the picture in the t-y plane showing constant solutions, and the regions in which the solutions go up or down. Show one solution for each of these regions.

Solution: (a) The equilibrium points are $y = 0$ and $y = \alpha$. If $\alpha > 0$, the right-hand side $y^3(\alpha - y)$ is negative for $y < 0$, positive for $0 < y < \alpha$, and negative again for $y > \alpha$. The picture of the phase line (drawn horizontally) is as follows:

$$\leftarrow \qquad 0 \qquad \rightarrow \qquad \alpha \qquad \leftarrow$$

$$\text{———————————}\bullet\text{—————}\bullet\text{————————}$$

The equilibrium point 0 is unstable, α asymptotically stable.

For $\alpha = 0$, the right-hand side is equal to $-y^4$, it is negative for all $y \neq 0$. The picture of the phase line (drawn horizontally) is such:

$$\leftarrow \qquad 0 \qquad \leftarrow$$

$$\text{———————————}\bullet\text{——————————}$$

The equilibrium point 0 is unstable (solutions starting at the points a little to the left of 0 go away from this point).

Finally, for α negative, the right-hand side is negative for $y < \alpha$ and for y positive, and $y^3(\alpha - y) < 0$ for y between α and 0:

$$\leftarrow \qquad \alpha \qquad \rightarrow \qquad 0 \qquad \leftarrow$$

$$\text{—————————}\bullet\text{—————}\bullet\text{——————————}$$

Here α is an unstable equilibrium point, 0 asymptotically stable.

(b) I cannot really draw pictures here, I'll describe the picture with words.

The picture for $\alpha > 0$ (say, for $\alpha = 4$) is like this: there are two constant solutions, drawn with horizontal lines, namely, $y(t) \equiv \alpha$ and $y(t0 \equiv 0$. In the region below the t-axis the solutions go down, and also in that above the line $y = \alpha$; while between the two horizontal lines $y = 0$ and $y = \alpha$ the solutions go up. For all solutions between these lines $\lim_{t\to\infty} y(t) = \alpha$, $\lim_{t\to-\infty} y(t) = 0$: the solution curves have two different horizontal asymptotes in different directions. For solutions with $y(t) > \alpha$ or $y(t) < 0$, the solution curves have only one horizontal asymptote: say, for $y(t) > \alpha$ we have $\lim_{t\to\infty} y(t) = \alpha$. As for $y(t) < 0$, we have a horizontal asymptote when we go to the left: $\lim_{t\to-\infty} y(t) = 0$; as t grows, $y(t)$ goes to $-\infty$ – but we don't know whether the interval of existence has ∞ as its right end, and $\lim_{t\to\infty} y(t) = -\infty$, or the interval of existence is some $(-\infty, b)$, and $\lim_{t\to b^-} y(t) = -\infty$, making $t = b$ a *vertical* asymptote of the solution curve.

We can answer this question easily enough by solving our separable equation – but we can draw a solution curve without this.

3 Is the differential equation

$$\frac{dy}{dt} = -\frac{y}{t} + 2$$

linear? Is it a linear homogeneous equation? It this equation an autonomous one?

Find the general solution of this equation (i. e., a formula, or formulas, for all solutions of this equation). Find its solution satisfying the initial condition $y(1) = 3$. What is the interval of existence of this solution?

Solution. Yes, our differential equation is linear, being of the form $\frac{dy}{dt} = a(t) \cdot y + f(t)$ with $a(t) = -\frac{1}{t}$ and $f(t) = 2$; and it is not linear homogeneous because $f(t) = 2 \not\equiv 0$. The equation is not autonomous because $a(t) \neq \text{const}$.

The integrating factor $u(t)$ can be obtained as

$$u(t) = e^{-\int a(t)\,dt} = e^{\int (1/t)\,dt} = e^{\ln |t|} = |t|,$$

that is, $u(t) = t$ for $t > 0$, and $u(t) = -t$ for $t < 0$ (for $t = 0$ the equation does not make any sense). An integrating factor can be obtained also as this function multiplied by any constant, in particular, by 1 for t positive, and by -1 for negative (once again, for $t = 0$ the equation does not make sense); that is, we can take as an integrating factor

$$u(t) = t \qquad \text{for } t > 0 \text{ and for } t < 0.$$

Now we take the term $a(t) \cdot y$ to the left-hand side (changing its sign), and multiply the equation by the integrating factor:

$$t \cdot \big(y' + \frac{1}{y} \cdot y\big) = ty' + y = 2t, \qquad \frac{d}{dt}\big(t \cdot y(t)\big) = 2t.$$

Finding easily the anti-derivative, we get:

$$t \cdot y(t) = \int 2t\,dt = t^2 + C, \qquad y(t) = t + \frac{C}{t},$$

where C is an arbitrary constant.

This is the general solution of our equation.

Now we take into account the initial condition:

$$y(1) = 1 + \frac{C}{1} = 3,$$

from which we have $C = 2$,

$$y(t) = t + \frac{C}{t}. \tag{3_1}$$

What is the interval of existence of this solution?

The expression (3_1) is defined, and satisfies our equation, for all $t \neq 0$, but we remember that our equation absolutely does not make sense for $t = 0$, which leaves two intervals: $(-\infty, 0)$ and $(0, \infty)$. Which of them?

Of course, we take the interval that contains the t-point $t = 1$ at which the initial condition was prescribed: i. e. the interval $(0, \infty)$.

So: the interval of existence is $(0, \infty)$.

4a On the moon colony, there are 5 births and 6 deaths per thousand people per year. One thousand people join the colony each month. The colony begins with zero population in the year 2012.

Denoting by $P(t)$ the population of the colony at time t, write a differential equation for $P(t)$ (be sure that you know in what units you measure time).

What is the order of this equation? Is it a linear or a non-linear equation?

With what initial condition(s?) should it be solved?

What will the population of the colony be in the year 3012?

Solution. We take the year 2012 as the time $t = 0$, and measure time in years. The equation for $P(t)$ is

$$\frac{dP}{dt} = 0.005 \cdot P(t) - 0.006 \cdot P(t) + 12,000 = -\,0.001 P(t) + 12,000$$

$(12,000$ people join the colony per *year*$)$.

This is a first-order equation; it is linear (with $a(t) \equiv -\,0.001$ and $f(t) \equiv 12,000$), not linear homogeneous.

The equation should be solved with the initial condition $P(0) = 0$ (the colony just starts in the year 2012, at $t = 0$); we have to find $P(1000)$, the population in the year 3012.

Let us solve the equation. The integrating factor is

$$u(t) = e^{-\int a(t)\,dt} = e^{\int 0.001\,dt} = e^{0.001\,t};$$

after taking the term with just $P(t)$ to the left-hand side and multiplying it by the integrating factor we get:

$$e^{0.001\,t}\big(P'(t) + 0.001 P(t)\big) = \frac{d}{dt}\big(e^{0.001\,t} \cdot P(t)\big) = 12,000\, e^{0.001\,t},$$

$$e^{0.001\,t} \cdot P(t) = \int 12,000\, e^{0.001\,t}\,dt = 12,000,000\, e^{0.001\,t} + C,$$

$$P(t) = 12,000,000 + C\, e^{-0.001\,t},$$

where C is an arbitrary constant.

This is the general solution of our equation.

Now we take into account the initial condition:

$$P(0) = 0 = 12,000,000 + C, \qquad C = -\,12,000,000$$

$$P(t) = 12,000,000 - 12,000,000\, e^{-0.001\,t}.$$

For $t = 1000$ we have:

$$P(1000) = 12,000,000 \cdot (1 - e^{-1}) \approx 7,585,400.$$

So this is the expected population of the colony in the year 3012.

4b A body having the temperature 100 degrees at the time $t = 0$ is placed into a room, where the temperature is 70 degrees. Write the differential equation for the temperature $T(t)$ of the body after time t. With what initial condition should this equation be solved?

The measurement shows that after 2 hours the temperature of the body is 85 degrees. The equation for the temperature contains a parameter that characterizes how fast the cooling goes. Use the measurement result to determine the value of this parameter.

What will the temperature of the body be 3 hours after it is placed into the room?

Solution. According to Newton's law of cooling (see the textbook, p. 31), we have:

$$\frac{dT}{dt} = -\,k(T - T_0),$$

where T_0 is the temperature of the surrounding medium (the ambient temperature, as the authors of the textbook formulate it). We have $T_0 = 70$ (70°), so

$$\frac{dT}{dt} = -\,k(T - 70).$$

This equation should be solved with the initial condition $T(0) = 100$.

We don't know the constant k, but let us solve the equation leaving the constant k as a letter: the integrating factor

$$u(t) = e^{kt},$$

the equation, after taking the term with the unknown function to the left-hand side and multiplying by $u(t)$:

$$e^{kt}(T' + kT) = \frac{d}{dt}\big(e^{kt} \cdot T(t)\big) = 70k \cdot e^{kt},$$

$$e^{kt} \cdot T(t) = 70\, e^{kt} + C, \qquad T(t) = 70 + C e^{-kt},$$

where C is an arbitrary constant. This is the general solution of our equation.

Taking the initial condition into account, we get:

$$T(0) = 100 = 70 + C, \qquad C = 30,$$

$$T(t) = 70 + 30\, e^{-kt}.$$

We don't know the constant k, but taking our measurement at the time $t = 2$ into account, we obtain:

$$T(2) = 85 = 70 + 30 \cdot e{-}k \cdot 2, \qquad 30\, e^{-2k} = 15, \qquad k = -\frac{\ln(15/30)}{2} = \frac{\ln 2}{2},$$

$$T(t) = 70 + 30\, e^{-(\ln 2/2)\cdot t}.$$

At $t = 3$ this is

$$T(3) = 70 + 30 \cdot e^{-(3/2)\ln 2} = 70 + 30 \cdot 2^{-3/2} \approx 80.6.$$

So the temperature after 3 hours will be approximately 80.6 degrees.

5a For the second-order differential equation

$$\frac{d^2y}{dt^2} = 2\frac{dy}{dt} + 3y: \qquad (*)$$

(a) Write a system of two first-order differential equations with two unknowns that is equivalent to the second-order equation $(*)$, taking $v = dy/dt$;
(b) Find the fundamental set of solutions of equation $(*)$, and write the general solution of this equation;
(c) Find the particular solution of equation $(*)$ satisfying the initial conditions $y(0) = 0,\ y'(0) = 1$;
(d) For this particular solution, plot $\big(y(t),\ v(t) = dy/dt\big)$, $-\infty < t < \infty$, in the y-v-plane. Show the direction of the motion along the curve you drew.

Solution. (a) The system is

$$\begin{cases} \dfrac{dy}{dt} = v, \\ \dfrac{dv}{dt} = 3y + 2v. \end{cases}$$

Using MATLAB, command `pplane` (`pplane7` for my computer) allows to draw the direction field corresponding to the system, and its solutions.

(b) Putting the function $y(t) = e^{\lambda t}$ into our equation, we see that this function is its solution if and only if the characteristic equation

$$\lambda^2 = 2\lambda + 3, \qquad \lambda^2 - 2\lambda - 3 = 0 \qquad (5a_1)$$

is satisfied (I am not repeating the calculations given in the lectures or in the textbook, p. 150). Solve equation $(5a_1)$:

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+12}}{2}, \qquad \lambda_1 = -1,\ \lambda_2 = 3.$$

So we have two solutions:

$$y_1(t) = e^{\lambda_1 t} = e^{-t}, \qquad y_2(t) = e^{\lambda_2 t} = e^{3t}.$$

These solutions are clearly not proportional to one another; they form a fundamental set of solutions.

The general solution of the equation $(*)$ is

$$y(t) = C_1 e^{-t} + C_2 e^{3t}.$$

(c) Taking the initial conditions into account:

$$y(0) = 0 = C_1 + C_2, \qquad y'(t) = -C_1 e^{-t} + 3C_2 e^{3t}, \quad y'(0) = 1 = -C_1 + 3C_2,$$

$$\begin{cases} C_1 + C_2 = 0, \\ -C_1 + 3C_2 = 1; \end{cases}$$

Solving this system, we get: $C_1 = -1/4$, $C_2 = 1/4$,

$$y(t) = -\frac{1}{4}\, e^{-t} + \frac{1}{4}\, e^{3t}.$$

(d) We have:

$$v(t) = \frac{dy}{dt} = \frac{1}{4}\, e^{-t} + \frac{3}{4}\, e^{3t}.$$

I cannot draw the pictures here, I'll try to describe it with words. As $t \to \infty$, the summands with e^{-t} go to 0, while $e^{3t} \to \infty$; the solution curve approaches the line described by $y(t) = \frac{1}{4}\, e^{3t}$ $v(t) = \frac{3}{4}\, e^{3t}$. This is a straight half-line going from the origin $(0,\, 0)$ to the first quadrant, with slope equal to 3. Our solution will have this half-line as its asymptote (as $t \to \infty$). As $t \to -\infty$, the summands with e^{3t} will go to 0, and the curve will approach as its asymptote the straight half-line described, parametrically, by $y(t) = -\frac{1}{4}\, e^{-t}$, $v(t) = \frac{1}{4}\, e^{-t}$: the half-line going from the origin into the second quadrant at a slope -1.

We described here what happens with $\big(y(t),\, v(t)\big)$ as t goes to $\pm\infty$; for $-\infty < t < \infty$, the point $\big(y(t),\, v(t)\big)$ will be within the angle with the asymptotes as its sides, up from the origin; the curve will be not unlike a branch of a hyperbola. The direction of the motion will be, roughly speaking, from left to right (put one or two arrows on the curve in your picture).

5b The hyperbolic cosine and sine are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Are the functions $f_1(x) = (\cosh x)^2$, $f_2(x) = (\sinh x)^2$, $f_3(x) = 1$ linearly independent or linearly dependent?

Don't write simply **yes** or **no** as an answer: it must have some support.

Solution. We have:

$$f_1(x) = (\cosh x)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}, \qquad f_2(x) = (\sinh x)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4},$$

so $f_1(x) - f_2(x) = f_3(x)$, $1 \cdot f_1(x) + (-1) \cdot f_2(x) + (-1) \cdot f_3(x) \equiv 0$, the linear combination of these functions with coefficients 1, -1, -1 is the identical zero; so the functions f_1, f_2, f_3 are linearly *dependent* (note the difference with the case of *two* functions: they are linearly dependent without any of them being proportional to another one).