WYTHOFF'S SEQUENCE AND N-HEAP WYTHOFF'S CONJECTURES

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ABSTRACT. Define a Wythoff's sequence as a sequence of pairs of integers $\{(A_n, B_n)\}_{n>n_0}$ such that there exists a finite set of integers T, $A_n = \text{mex}(\{A_i, B_i : i < n\} \cup T)$, $B_n - A_n = n$, and $\{B_n\} \cap T = \emptyset$. Structural properties and behaviors of Wythoff's sequence are investigated. The main result is that for such a sequence, there always exists an integer α such that when n is large enough, $|A_n - \lfloor n\phi \rfloor - \alpha| \leq 1$, where $\phi = (1+\sqrt{5})/2$, the golden section. The value of α can also be easily determined by a relatively small number of pairs in the sequence. As a corollary, the two conjectures on the N-heap Wythoff's game by Fraenkel [3] on the N-heaped Wythoff's game are proved to be equivalent.

1. Introduction

Wythoff's pairs are pairs of integers $\{(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)\}_{n\geq 0}$, where throughout this paper, $\phi = (1+\sqrt{5})/2$, is the golden section. The first few pairs are listed in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$A_n = \lfloor n\phi \rfloor$	0	1	3	4	6	8	9	11	12	14	16	17	19	21
$B_n = \lfloor n\phi^2 \rfloor$	0	2	5	7	10	13	15	18	20	23	26	28	31	34

Wythoff's pairs have close relationships with the Fibonacci numbers. For example, let us consider the sequence A_1, B_1, A_{B_1}, B_{B_1}, $A_{B_{B_1}}$, $B_{B_{B_1}}$, ... This is the Fibonacci sequence without the first term. In fact, any such sequence starting from A_n and B_n is a Fibonacci sequence generated by those two integers, as proved by Hoggatt and Hillman [8], Horadam [9], and Silber [10]. Other properties, relationships and applications were investigated extensively by numerous people, whom we are not going to list here.

Wythoff's pairs were first found as the result of a mathematical game [12]: the game consists of two piles of tokens. Two players alternate in removing any number of tokens from a single pile: at least one, and up to the whole pile; or removing the *same* positive number of tokens from both piles. The first player who cannot make a move loses. Wythoff's pairs can therefore be interpreted as $\{A_n, B_n\}_{n\geq 0}$, such that $A_n = \text{mex}\{A_m,\ B_m\ :\ 0 \leq m < n\}$ and $B_n = A_n + n$. Recall that for a set of non-negative integers S, $\text{mex}(S) = \min(\mathbb{Z}_{\geq 0} - S)$ is the *M*inimal *EX*clusive value of S, i.e., the least nonnegative integer that is *not* in the set S. For example,

$$\text{mex}(\{0, 1, 2, 4, 7, 8\}) = 3.$$

Note that $A_0 = B_0 = 0$. The winning strategies were described by Fraenkel [4], and also in WW [2]. Periodic properties of the Sprague-Grundy function and other generalizations of the game were also discussed. Please see the manuscript by Fraenkel [3] for the complete list of the progress.

Another elegant generalization of the game involving more than two piles was proposed by Fraenkel [3], which is also listed in the survey article by Guy and Nowakowski [6] as one of the "unsolved problems in combinatorial games": given N piles of tokens, whose sizes are $A^1, \ldots, A^N$, $A^1 \le \cdots \le A^N$. A player can remove any number of tokens from a single pile, or remove $(a_1, \ldots, a_N)$ tokens from all piles — a_i tokens from the i-th pile, providing that $0 \le a_i \le A^i$, $\sum_{i=1}^N a_i > 0$, and $a_1 \oplus \cdots \oplus a_N = 0$, where $\oplus$ is the nim addition (XOR binary operation). Denote all the P-positions by $(A^1, \ldots, A^{N-2}, A_n^{N-1}, A_n^N)$, $A^{N-2} \le A_n^{N-1} \le A_n^N$ and $A_n^{N-1} < A_{n+1}^{N-1}$ for all $n \ge 0$. Two conjectures were presented on the game, when $A^1, \ldots, A^{N-2}$ are fixed:

Conjecture 1: There exists an integer N_1 such that when $n > N_1$, $A_n^N = A_n^{N-1} + n$.

Conjecture 2: There exist integers N_2 and α_2 such that when $n > N_2$, $A_n^{N-1} = \lfloor n\phi \rfloor + \epsilon_n + \alpha_2$ and $A_n^N = A_n^{N-1} + n$, where $-1 \le \epsilon_n \le 1$.

Furthermore, $A_n^{N-1} = \text{mex}(\{A_i^{N-1}, A_i^N : 0 \le i < n\} \cup T)$, where T is a small set of integers.

Recall that P-positions of the original Wythoff's game can be written as $\{\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor\}$, and observe that the winning strategy of the multiple-heap Wythoff's game is also related to the golden section. This generalization of the Wythoff's game is by far the only one that can do so. Doron Zeilberger and the author [11] proved the conjectures for the three-heap game when the first heap has up to 10 tokens.

The structure of the paper follows.

- In Section 2 we are going to discuss the definition of Wythoff's sequence and an alternative way of constructing and verifying a Wythoff's sequence in Theorem 2.2;
- In Section 3 we are to discuss some essential properties of Wythoff's sequence. Theorem 3.1 and its corollaries reveal that after some chaotic beginning, each Wythoff's sequence is well organized.
- In Section 4 we prove the equivalency of the two kinds of Wythoff's sequence in Theorem 4.1 and its implication on Fraenkel's conjectures on N-heap Wythoff's game. In fact, as shown in Corollary 4.6, The first conjecture implies the second. We further analyze the behavior of $\alpha + \epsilon_n$ as defined in Conjecture 2 in the section, and conclude in Theorem 4.7 that α is an intrinsic property of a Wythoff's sequence that can be evaluated with a small set of data, instead of using limit as implied in the conjecture.

2. Wythoff's Sequence

Definition 1. We call a sequence of pairs of integers $\{(A_n, B_n)\}_{n\geq n_0}$ a *Wythoff's sequence* if $n_0 > 0$ and there exists a finite set of integers T such that $A_n = \text{mex}(\{A_i, B_i : n_0 \leq i < n\} \cup T)$, $B_n = A_n + n$ and $\{B_n\} \cap T = \emptyset$.

Definition 2. A *special Wythoff's sequence* is a Wythoff's sequence such that there exist integers N and α such that when $n > N$, $A_n = \lfloor n\phi \rfloor + \alpha + \epsilon_n$, where $\epsilon_n \in \{0, \pm 1\}$.

Lemma 2.1. *For a Wythoff's sequence $\{(A_n, B_n)\}_{n\geq n_0}$, let N be an integer such that when $n \geq N$, $A_n > \max(T)$, where T is given in Definition 1. Then*

(1) $1 \leq A_{n+1} - A_n \leq 2$,
(2) $2 \leq B_{n+1} - B_n \leq 3$, *and*
(3) *if* $A_{n+1} - A_n = 1$, *then* $A_{n+2} - A_{n+1} = A_n - A_{n-1} = 2$,
(4) $|\lfloor n_1\phi \rfloor - \lfloor n_2\phi \rfloor - (n_1 - n_2)\phi| < 1$.

Proof. The proof appears in [11]. For completeness of the paper, we provide it here too.

First since T is finite, the integer N definitely exists. Secondly, since A_n is defined using the mex function, each A_n is the smallest available in $\mathbb{Z}_{\geq 0} - T - \{A_i, B_i | i < n\}$. So $\{A_n\}$ is an increasing sequence. At the same time, $B_n = A_n + n$ must be increasing with n too. Thirdly, if an integer is not in $\mathbb{Z}_{\geq 0} - T - \{B_n\}$, it eventually must be in $\{A_n\}$, i.e., any nonnegative integer not in T must be either an A or B.

(1) If $A_{n+1} - A_n \geq 2$, since $A_n + 1$ and $A_n + 2$ are not A's, they must be B's. In other words, there exist distinct integers m_1 and m_2 such that $B_{m_1} = A_n + 1$ and $B_{m_2} = A_n + 2$. Therefore $A_{m_1} - A_{m_2} = m_2 - m_1 - 1$. Since $\{A_n\}$ is increasing, the two sides of the last equation cannot have the same signs. A contradiction.
(2) For each n, $B_{n+1} - B_n = (A_{n+1} + n + 1) - (A_n + n)$. Hence item 2 follows.
(3) If $A_{n+2} - A_{n+1} = A_{n+1} - A_n = 1$, let m be the largest such that $B_m < A_n$. Then $B_{m+1} - B_m > 3$, which is contradictory to item 2. The other half of item 3 can be proved similarly.
(4) $-1 < \lfloor n_1\phi \rfloor - n_1\phi < \lfloor n_1\phi \rfloor - \lfloor n_2\phi \rfloor - (n_1 - n_2)\phi < -\lfloor n_2\phi \rfloor + n_2\phi < 1$.

□

From now on, we always assume n_0 is an integer such that the above Lemma is satisfied for all $n > n_0$. Otherwise, we can always remove a few pairs of the sequence at the beginning.

The above definitions only provide ways to *verify* a given sequence is a Wythoff's sequence or a special Wythoff's sequence. The following theorem provides a way to *create* a Wythoff's sequence.

Theorem 2.2. *Given two finite sets of integers S_1 and $S_2 \subset \mathbb{Z}_{\geq 0}$, define a sequence $\{A_n, B_n\}$ such that*

$$A_n = \operatorname{mex}(\{A_i, B_i\}_{i<n} \cup S_1), \tag{2.1}$$
$$B_n = \operatorname{mex}(\{A_i, B_i\}_{i<n} \cup S_1 \cup \{A_n + m : m \in S_2 \text{ or } m \in \{B_i - A_i\}_{i<n}\}), \tag{2.2}$$

then eventually $\{(A_n, B_n)\}$ will be a Wythoff's sequence. Conversely, given a Wythoff's sequence there exist two finite sets S_1 and S_2 that satisfy the conditions above.

Proof. If we have finite sets S_1 and S_2 that satisfy 2.1 and 2.2, we want to prove that eventually $B_{n+1} - A_{n+1} = B_n - A_n + 1$ and $T = S_1$, where T is given in Definition 1. After chopping off some pairs at the beginning and reorganizing the indices, the sequence $\{(A_n, B_n)\}$ will be a Wythoff's sequence.

Since both A_n and B_n are defined using the mex function, we can tell that each A_n is the smallest number available that is *not* in S_1 and *not* a previously found A or B. Similarly each B_n is the smallest available number such that it is not in S_1, not a previous A or B. At the mean time $B_n - A_n$ cannot be in S_2 or $\{B_i - A_i\}_{i<n}$.

Thus we know that

$$|\{A_n\}_{n\geq 0} \cap \{B_n\}_{n\geq 0}| \leq 1,$$

where the equal sign can hold if and only if $0 \notin S_2$;

$$B_n \geq A_n;$$

$$\{A_n\}_{n\geq 0} \cup \{B_n\}_{n\geq 0} = \mathbb{Z}_{\geq 0} - S_1,$$

since A_n is always the smallest number available, any nonnegative integer not in S_1 must be in either $\{A_n\}$ or $\{B_n\}$; and

$$\{B_n - A_n\}_{n\geq 0} \cap S_2 = \emptyset.$$

Furthermore

$$A_m \neq A_n, \ B_m \neq B_n, \text{ and } B_m - A_m \neq B_n - A_n$$

for any $m \neq n$. So

$$T = \mathbb{Z}_{\geq 0} - \{B_i, A_i : i \geq 0\} = S_1,$$

which is finite.

Since no two A_n's are the same and S_1 is finite, eventually A_n will be greater than $\max(S_1)$. Similarly $B_n - A_n$ will eventually be greater than $\max(S_2)$, i.e., there exists N_0 such that when $n \geq N_0$, $B_n > A_n > \max(S_1)$ and $B_n - A_n > \max(S_2)$. By the definition of A_n, which is always the smallest number available, $\{A_n\}$ is increasing with n. Here we are not saying $\{B_n - A_n\}$ is increasing with n, although the result of the theorem clearly indicates they will be.

Define $\alpha_n = \max\{B_i - A_i : i < n\} + 1$ and $D_n = \{i : 0 \le i < \alpha_n\} - S_2 - \{B_i - A_i : i < n\}$. For any $n \ge N_0$,

$$\begin{aligned} & B_n - A_n \\ = \;& \text{mex}(\{A_i, B_i\}_{i<n} \cup S_1 \cup \{A_n + m : m \in S_2 \text{ or } m \in \{B_i - A_i\}_{i<n}\}) - A_n \\ = \;& \text{mex}(\{\{A_i - A_n, B_i - A_n\}_{i<n} \cup \{m : m \in S_2 \text{ or } m \in \{B_i - A_i\}_{i<n}\}\} \cap \mathbb{Z}_{\ge 0}) \\ \le \;& \max(\{B_i - A_n\}_{i<n} \cup \{m : m \in S_2 \text{ or } m \in \{B_i - A_i\}_{i<n}\}) + 1 \\ \le \;& \max(\{B_i - A_i\}_{i<n} \cup \{m : m \in \{B_i - A_i\}_{i<n}\}) + 1 \\ = \;& \max\{B_i - A_i : i < n\} + 1 \\ = \;& \alpha_n. \end{aligned} \tag{2.3}$$

It is also clear that $\alpha_n \le \alpha_{n+1}$. Now consider the following:

If $B_n - A_n < \alpha_n$, then $B_n - A_n \in D_n$ by the definition of D_n;
If $B_n - A_n \in D_n$, then $B_n - A_n < \alpha_n$, so $\alpha_{n+1} = \alpha_n$;
If $\alpha_{n+1} = \alpha_n$, then $\{B_n - A_n\} < \alpha_{n+1} = \alpha_n$, hence $D_n = D_{n+1} \cup \{B_n - A_n\}$;
If $D_n = D_{n+1} \cup \{B_n - A_n\}$, then $\{B_n - A_n\} \in D_n$, therefore $B_n - A_n < \alpha_n$.

Comparing the first and the last of the above arguments, we know that all of the conditions are equivalent. Similarly,

$B_n - A_n = \alpha_n$ iff $\alpha_{n+1} = \alpha_n + 1$; iff $D_{n+1} = D_n$.

By 2.3, $B_n - A_n \le \alpha_n$, so $D_{n+1} \subset D_n$ as proved in either of the two cases above. Now that $D_{n+1} \subset D_n \subset D_{N_0}$ are all finite, there exists $N \ge N_0$ such that for any $n \ge N$, $D_n = D_{n+1}$, thus $B_{n+1} - A_{n+1} = \alpha_{n+1} = \alpha_n + 1 = B_n - A_n + 1$.

Conversely, if $\{(A_n, B_n)\}$ is a Wythoff's sequence, we can define $S_1 = \mathbb{Z}_{\ge 0} - \{A_i, B_i : i > 0\}$ and $S_2 = \mathbb{Z}_{\ge 0} - \{B_i - A_i : i > 0\}$, which are both finite by the definition of the Wythoff's sequence. □

Corollary 2.3. *Given $\{(A_n, B_n)\}_{n \ge 0}$, S_1 and S_2 as in Theorem 2.2, we have*

$$S_2 = \mathbb{Z}_{\ge 0} - \{B_i - A_i : i > 0\}. \tag{2.4}$$

Proof. First observe that $S_2 \subset \mathbb{Z}_{\ge 0} - \{B_i - A_i : i > 0\}$ by the definition of B_n. Secondly if there exists $d \in \mathbb{Z}_{\ge 0} - \{B_i - A_i : i > 0\} - S_2$, we know $B_n \ne A_n + d$ for all n. Choose N large enough so that $B_i - A_i > d$ for all $i \ge N$. Since for any $n \ge N$, $(A_n, A_n + d) \notin \{(A_i, B_i) : i > 0\}$. The only reasons that this can happen are:

- $A_n + d \in \{A_i\}_{0<i<n}$, but this is impossible because $\{A_i\}_{0<i<n}$ is an increasing sequence;
- $B_i - A_i = d$ for some $i < n$, but this is impossible because of our assumption on d;
- there exists an m such that $B_m = A_n + d$, and this is the only possibility.

Therefore for any $n \ge N$, there exist m_1 and m_2 such that $A_{n+1} - A_n = B_{m_1} - B_{m_2}$. By Lemma 2.1, $A_{n+1} - A_n = 2 = B_{m_1} - B_{m_2}$, thus $B_{n+1} - B_n = A_{n+1} - A_n + 1 = 3$ for all $n \ge N$.

n	A_n	B_n	$B_n - A_n$
	1	1	0
	3	5	2
	6	12	6
	8	9	1
	11	15	4
	13	20	7
	14	23	9
11	16	27	11
12	17	29	12
13	18	31	13

n	A_n	B_n	$B_n - A_n$
14	19	33	14
15	21	36	15
16	22	38	16
17	24	41	17
18	25	43	18
19	26	45	19
20	28	48	20
21	30	51	21
22	32	54	22
23	34	57	23

n	A_n	B_n	$B_n - A_n$
24	35	59	24
25	37	62	25
26	39	65	26
27	40	67	27
28	42	70	28
29	44	73	29
30	46	76	30
31	47	78	31
32	49	81	32
33	50	83	33

Table 1. Example of Theorem 2.2

If this is true, $A_{3n} - A_n = 2(3n-n) = 4n$, while $B_{2n} - A_n = B_{2n} - B_n + n = 3(2n-n) + n = 4$. So $A_{3n} = B_{2n}$ for all $n \geq N$, which is contradictory to the fact that $|\{A_n\} \cap \{B_n\}| \leq 1$ as discussed in the proof of Theorem 2.2. □

Example 1. Let $S_1 = \{0, 2, 4, 7, 10\}$, $S_2 = \{3, 5, 8, 10\}$ and define $\{A_n, B_n\}$ as in Theorem 2.2. The first 30 pairs are shown in Table 1. We can convince ourselves that the sequence becomes a Wythoff's sequence starting from the 8-th pair, and the set of numbers not in $\{B_n - A_n\}$ is S_2. After chopping off the first 7 pairs and reorganize the indices, we can make $n = B_n - A_n$.

So even though we can start with two random finite sets of integers S_1 and S_2, and define $\{A_n, B_n\}$ as in Theorem 2.2, after some chaotic data at the beginning, the sequence defined using mex will eventually grow in an orderly manner, and become a Wythoff's sequence.

3. Properties of Wythoff's Sequence

In this section, we are going to explore some basic properties of Wythoff's sequence by (literally) counting the numbers.

Theorem 3.1. *For any Wythoff's sequence $\{(A_n, B_n)\}_{n \geq n_0}$, there exist constants N and c, such that when $n \geq N$, $A_{A_n+c} = B_n - 1$ and $A_{B_n+c} = B_{A_n+c} + 1 = A_n + B_n + c$.*

Proof. By Lemma 2.1, the difference between any two consecutive B's is at least 2, and any integer must be either an A, B or T. Choose N so that when $n \geq N - 1$, any integer falls in between two B's is in A. Therefore there exists k_0 such that $A_{k_0} = B_N - 1$.

Counting all the integers from 1 to $B_N - 1$, we have

- $(k_0 - n_0 + 1)$ A's, i.e., $A_{n_0}, \ldots, A_{k_0}$;
- $(N - n_0)$ B's, i.e., $B_{n_0}, \ldots B_{N-1}$; and
- $|T|$ T's,

which indicates $B_N - 1 = k_0 - n_0 + 1 + N - n_0 + |T|$. Letting

$$c = k_0 - A_N, \tag{3.1}$$

we have

$$|T| = B_N - 1 - k_0 + n_0 - 1 - N + n_0 = A_N + 2n_0 - k_0 - 2 = 2n_0 - 2 - c. \tag{3.2}$$

Similarly for any $n \geq n_0$, there exists $A_k = B_n - 1$. Counting all the integers from 1 to B_n, we have

- $(k - n_0 + 1)$ A's, i.e., $A_{n_0}, \ldots, A_k$,
- $(n - n_0 + 1)$ B's, i.e., $B_{n_0}, \ldots, B_n$, and
- $|T|$ T's,

so $B_n = k - n_0 + 1 + n - n_0 + 1 + |T| = k + n - c$, hence $k = B_n - n + c = A_n + c$. Therefore $B_n = A_k + 1 = A_{A_n+c} + 1$.

Counting all the integers from 1 to B_{A_n+c}, we have

- $(A_n + c - n_0 + 1)$ B's, i.e., $B_{n_0}, \ldots, B_{A_n+c}$, and
- $|T|$ T's,

so there are $(B_{A_n+c} - A_n - c + n_0 - 1 - |T|) = (A_{A_n+c} + n_0 - 1 - |T|)$ A's. The largest of these is $A_{k'} = B_{A_n+c} - 1$. Since the indices start from n_0, k' must be $(A_{A_n+c} + n_0 - 1 - |T|) + (n_0 - 1) = B_n - 1 + 2n_0 - 2 - (2n_0 - 2 - c) = B_n + c - 1$. By Lemma 2.1 and the previous result, $A_{B_n+c} = A_{k'+1} = B_{A_n+c} + 1 = A_{A_n+c} + A_n + c + 1 = A_n + B_n + c$. □

Corollary 3.2. *Given a Wythoff's sequence $\{(A_n, B_n)\}_{n \geq n_0}$ and c as define in 3.1, we have*

1. $A_{A_n+c+1} - A_{A_n+c} = 2$;
2. $A_{B_n+c+1} - A_{B_n+c} = 1$;
3. $B_{A_n+c+1} - B_{A_n+c} = 3$;
4. $B_{B_n+c+1} - B_{B_n+c} = 2$.

Proof. $A_{m+c+1} - A_{m+c} = 2$ iff there exists n, such that $B_n = A_{m+c+1} - 1 = A_{m+c} + 1$; iff $A_{m+c} = A_{A_n+c}$, by Theorem 3.1; iff $m = A_n$.

Since all of the above conditions are equivalent, and by Lemma 2.1, $A_{B_n+c+1} - A_{B_n+c}$ can only be 1.

Therefore $B_{A_n+c+1} - B_{A_n+c} = (A_{A_n+c+1} + A_n + c + 1) - (A_{A_n+c} + A_n + c) = 3$, and $B_{B_n+c+1} - B_{B_n+c} = (A_{B_n+c+1} + B_n + c + 1) - (A_{B_n+c} + B_n + c) = 2$. □

Notice that if there exist $m_1 > m_2 > n_0$ such that $A_{m_1} \geq B_{m_2}$, and $\{(A_n, B_n) : m_2 \leq n \leq m_1\}$ are given, we can construct the rest of the sequence for $m > m_1$ without using the definition of the Wythoff's sequence, i.e., mex. There are two ways of doing so recursively:

(1) For any $m > m_1$, by Theorem 3.1, if $m - c$ is of the form $A_{m'}$, $A_m = A_{A_{m'}+c} = B_{m'} - 1 = A_{m'} + m' - 1$ and $B_m = A_m + m = m + B_{m'} - 1$; otherwise, $m - c = B_{m'}$, $A_m = A_{B_{m'}+c} = A_{m'} + B_{m'} + c = A_{m'} + m$ and $B_m = A_m + m = B_{m'} + 2m - m'$.
(2) Suppose A_m is known. If $m - c$ is in the A's, by Corollary 3.2, $A_{m+1} = A_m + 2$ and $B_{m+1} = B_m + 3$; otherwise, $A_{m+1} = A_m + 1$ and $B_{m+1} = B_m + 2$. Here we can see that the two sequences are self-generating, i.e., we can construct the sequence of either $\{A_n\}_{n \geq m_2}$ or $\{B_n\}_{n \geq m_2}$ without any knowledge of the other.

Corollary 3.3. *Given a Wythoff's sequence $\{(A_n, B_n)\}_{n \geq n_0}$, we have the following:*

(1) *for any $n \geq A_{n_0}$, the number of A's less than n is $A_{n+c} - n - n_0 + 1$;*
(2) *for any $n \geq B_{n_0}$, the number of B's less than n is $2n - A_{n+c} + c - n_0$.*

Proof. Let $f(n) = A_{n+c} - n - n_0 + 1$. We claim that $f(n)$ is the number of A's less than n. First $f(A_{n_0}) = A_{A_{n_0}+c} - A_{n_0} - n_0 + 1 = B_{n_0} - A_{n_0} - n_0 = 0$, which is the number of A's less than A_{n_0}. By induction, if the claim is true for $n - 1$, there are two cases:

- if $n - 1 = B_m$, the number of A's less than n should be the same as that of $n - 1$. By Corollary 3.2, $f(n) = A_{B_m+1+c} - (B_m + 1) - n_0 + 1 = A_{B_m+c} - B_m - n_0 + 1 = f(n-1)$;
- if $n - 1 = A_m$, the number of A's less than n should be one plus that of $n - 1$. Meanwhile $f(n) = A_{A_m+1+c} - (A_m + 1) - n_0 + 1 = A_{A_m+c} + 2 - A_m - n_0 = f(n-1) + 1$.

So the claim is proved.

On the other hand, if we write $g(n) = 2n - A_{n+c} + c - n_0$, we claim that $g(n)$ is the number of B's less than n. Like the proof above, we only need to prove the following:

- $g(B_{n_0}) = 2B_{n_0} - A_{B_{n_0}+c} + c - n_0 = 2B_{n_0} - A_{n_0} - B_{n_0} - c + c - n_0 = 0$;
- if $n - 1 = B_m$, $g(n) = 2n - A_{B_m+1+c} - n_0 = 2n - A_{B_m+c} - 1 - n_0 = g(n-1) + 1$;
- if $n - 1 = A_m$, $g(n) = 2n - A_{A_m+1+c} - n_0 = 2n - A_{A_m+c} - 2 - n_0 = g(n-1)$.

Thus the proof is completed. □

Example 2. Let us examine the Wythoff's sequence shown in Example 1. Since the sequence starts with index 11, $n_0 = 11$, $A_{n_0} = 16$, $B_{n_0} = 27$, and $A_{19} = 26 = B_{n_0} - 1 = A_{A_{n_0}+c} = A_{16+c}$ by Theorem 3.1, so $c = 3$. We can also easily check that when $n = 11$, $A_{B_n+c} = A_{30} = 46 = B_{19} + 1 = B_{A_n+c} + 1 = A_n + B_n + c$. The two corollaries can also be verified with small n's.

Example 3. A special case of the theorem and corollaries is when the Wythoff's sequence is the original Wythoff's pairs. In such an occasion, $n_0 = 0$ and $c = 0$, which were proved by Hoggatt, Hillman [8], Hoggatt, Bicknell-Johnson [7], and Silber [10].

4. Special Wythoff's Sequence and N-heap Wythoff's Conjectures

Throughout this section we use Wythoff's sequence $\{(A_n, B_n)\}_{n\geq n_0}$ and c as in 3.1. Note that when n is large enough, it must be of the form A_{A_m}, A_{B_m}, B_{A_m}, or B_{B_m}. Since for any m there exist m_1 and m_2 such that $A_m = B_{m_1}+1$ or $B_{m_1}-1$, and $B_m = A_{m_2}+1$ by Lemma 2.1, n must be of also the form $B_{A_{m+c+\epsilon_2}}+c+\epsilon_1$, where $\epsilon_1 \in \{-1,0,1\}$ and $\epsilon_2 \in \{0,1\}$, although the representation may not be unique.

Theorem 4.1. *Every Wythoff's sequence is special.*

Proof. Let $\alpha_n = A_n - \lfloor n\phi \rfloor$. By definition, we only need to prove that if m and n are large enough, $|\alpha_m - \alpha_n|$ is at most 2.

By Lemma 4.4 below, we know α_n is bounded, say, by M.

Define a function $\Delta(n) = B_{A_{n+c+1}}+c+1$.

For integers $m, n \geq \Delta^{2M-1}(n_0)$, we can construct two sequences $a_1, \dots, a_k$ and $b_1, \dots, b_k$, such that $k \geq 2M-1$, $a_k = m$, $b_k = n$, $A_{n_0} \leq \min(a_1, b_1) < B_{A_{n_0+c+1}}+c+1$, and $a_i = B_{A_{a_{i-1}+c+\epsilon_{a2}^{(i)}}}+\epsilon_{a1}^{(i)}$, $b_i = B_{A_{b_{i-1}+c+\epsilon_{b2}^{(i)}}}+\epsilon_{b1}^{(i)}$, where $1 < i \leq k$, $\epsilon_{a1}^{(i)}, \epsilon_{b1}^{(i)} \in \{0, \pm 1\}$, and $\epsilon_{a2}^{(i)}, \epsilon_{b2}^{(i)} \in \{0,1\}$. By Corollary 4.3 below, we know $|\alpha_{a_i} - \alpha_{b_i}| \leq \max(|\alpha_{a_{i-1}} - \alpha_{b_{i-1}}| - 1, 2)$. Hence

$$\begin{aligned}
& |\alpha_m - \alpha_n| \\
= & |\alpha_{a_k} - \alpha_{b_k}| \\
\leq & \max(|\alpha_{a_1} - \alpha_{b_1}| - (k-1), 2) \\
\leq & \max(2M - (2M-2), 2) \\
= & 2.
\end{aligned}$$

□

Now all we need to do is to prove the following lemmas and corollary.

Lemma 4.2. *If $-1 \leq \epsilon_{m1}, \epsilon_{n1} \leq 1$ and $0 \leq \epsilon_{m2}, \epsilon_{n2} \leq 1$, we have*

$$\alpha_{B_{A_{m+c+\epsilon_{m2}}}+c+\epsilon_{m1}} - \alpha_{B_{A_{n+c+\epsilon_{n2}}}+c+\epsilon_{n1}} = -(\alpha_m - \alpha_n)(2\phi - 3) + \xi,$$

with $|\xi| < 2$.

Furthermore $\xi = \xi_m - \xi_n$, where ξ_m depends only on m, m_1 and m_2, while ξ_n depends only on n, n_1 and n_2. $|\xi_m|, |\xi_n| < 1$.

Proof. By Corollary 3.2, $A_{B_{n+c+1}} - A_{B_{n+c}} - \phi = 1 - \phi$ and $A_{B_{n+c-1}} - A_{B_{n+c}} + \phi = -2 + \phi$, so $A_{B_{n+c+\epsilon}} - A_{B_{n+c}} - \phi\epsilon = (3\epsilon - 2\phi\epsilon - \epsilon^2)/2$ when $|\epsilon| \le 1$. Therefore if we write

$$\gamma = (A_{B_{m+c+\epsilon_m}} - A_{B_{m+c}} - \phi\epsilon_m) - (A_{B_{n+c+\epsilon_n}} - A_{B_{n+c}} - \phi\epsilon_n),$$

we have

$$|\gamma| = |(\epsilon_m - \epsilon_n)(3 - 2\phi - \epsilon_m - \epsilon_n)/2| \le \phi - 1,$$

when $|\epsilon_m|$, $|\epsilon_n| \le 1$. Also note that $A_{A_{n+c+\epsilon}} - A_{A_{n+c}} = 2\epsilon$, when $\epsilon \in \{0, 1\}$.

Define

$$\begin{aligned} \beta_1 &= \lfloor (B_{A_{m+c+\epsilon_{m2}}} + c + \epsilon_{m1})\phi \rfloor - \lfloor (B_{A_{n+c+\epsilon_{n2}}} + c + \epsilon_{n1})\phi \rfloor \\ &\quad -((B_{A_{m+c+\epsilon_{m2}}} + c + \epsilon_{m1})\phi - (B_{A_{n+c+\epsilon_{n2}}} + c + \epsilon_{n1})\phi) \end{aligned}$$

and

$$\beta_2 = \lfloor m\phi \rfloor - \lfloor n\phi \rfloor - (m - n)\phi.$$

By Lemma 4.4, we know

$$|\beta_1|, |\beta_2| < 1.$$

Now if $\epsilon_{m1}, \epsilon_{n1} \in \{-1, 0, 1\}$ and $\epsilon_{m2}, \epsilon_{n2} \in \{0, 1\}$,

$$\begin{aligned}
&\alpha_{B_{A_{m+c+\epsilon_{m2}}}+c+\epsilon_{m1}} - \alpha_{B_{A_{n+c+\epsilon_{n2}}}+c+\epsilon_{n1}} \\
=\ & A_{B_{A_{m+c+\epsilon_{m2}}}+c+\epsilon_{m1}} - A_{B_{A_{n+c+\epsilon_{n2}}}+c+\epsilon_{n1}} \\
& -(\lfloor (B_{A_{m+c+\epsilon_{m2}}} + c + \epsilon_{m1})\phi \rfloor - \lfloor (B_{A_{n+c+\epsilon_{n2}}} + c + \epsilon_{n1})\phi \rfloor) \\
=\ & A_{B_{A_{m+c+\epsilon_{m2}}}+c+\epsilon_{m1}} - A_{B_{A_{n+c+\epsilon_{n2}}}+c+\epsilon_{n1}} \\
& -((B_{A_{m+c+\epsilon_{m2}}} - B_{A_{n+c+\epsilon_{n2}}})\phi + (\epsilon_{m1} - \epsilon_{n1})\phi + \beta_1) \\
=\ & A_{B_{A_{m+c+\epsilon_{m2}}}+c} - A_{B_{A_{n+c+\epsilon_{n2}}}+c} + \gamma - (B_{A_{m+c+\epsilon_{m2}}} - B_{A_{n+c+\epsilon_{n2}}})\phi - \beta_1 \\
=\ & A_{A_{m+c+\epsilon_{m2}}} - A_{A_{n+c+\epsilon_{n2}}} + (B_{A_{m+c+\epsilon_{m2}}} - B_{A_{n+c+\epsilon_{n2}}})(1 - \phi) + \gamma - \beta_1 \\
=\ & (A_{A_{m+c+\epsilon_{m2}}} - A_{A_{n+c+\epsilon_{n2}}})(2 - \phi) + (A_m - A_n + \epsilon_{m2} - \epsilon_{n2})(1 - \phi) + \gamma - \beta_1 \\
=\ & (A_{A_{m+c}} - A_{A_{n+c}} + 2(\epsilon_{m2} - \epsilon_{n2}))(2 - \phi) + (A_m - A_n + \epsilon_{m2} - \epsilon_{n2})(1 - \phi) + \gamma - \beta_1 \\
=\ & (A_{A_{m+c}} - A_{A_{n+c}})(2 - \phi) + (A_m - A_n)(1 - \phi) + (\epsilon_{m2} - \epsilon_{n2})(5 - 3\phi) + \gamma - \beta_1 \\
=\ & (B_m - B_n)(2 - \phi) + (A_m - A_n)(1 - \phi) + (\epsilon_{m2} - \epsilon_{n2})(5 - 3\phi) + \gamma - \beta_1 \\
=\ & (A_m - A_n)(3 - 2\phi) + (m - n)(2 - \phi) + (\epsilon_{m2} - \epsilon_{n2})(5 - 3\phi) + \gamma - \beta_1 \\
=\ & (\lfloor m\phi \rfloor + \alpha_m - \lfloor n\phi \rfloor - \alpha_n)(3 - 2\phi) + (m - n)(2 - \phi) + (\epsilon_{m2} - \epsilon_{n2})(5 - 3\phi) + \gamma - \beta_1 \\
=\ & ((m - n)\phi + \beta_2 + (\alpha_m - \alpha_n))(3 - 2\phi) + (m - n)(2 - \phi) \\
& +(\epsilon_{m2} - \epsilon_{n2})(5 - 3\phi) + \gamma - \beta_1 \\
=\ & -(\alpha_m - \alpha_n)(2\phi - 3) - \beta_2(2\phi - 3) + (\epsilon_{m2} - \epsilon_{n2})(5 - 3\phi) + \gamma - \beta_1.
\end{aligned}$$

Let $\xi = -\beta_2(2\phi-3)+(\epsilon_{m2}-\epsilon_{n2})(5-3\phi)+\gamma-\beta_1$, then

$$\begin{aligned} & |\xi| \\ \leq\ & |\beta_2|(2\phi-3)+|\epsilon_{m2}-\epsilon_{n2}|(5-3\phi)+|\gamma|+|\beta_1| \\ <\ & (2\phi-3)+(5-3\phi)+\phi-1+1 \\ =\ & 2, \end{aligned}$$

Define

$$\begin{aligned} \xi_m &= -(\lfloor m\phi\rfloor - m\phi)(2\phi-3)+\epsilon_{m2}(5-3\phi)+\epsilon_m(3-2\phi)/2-\epsilon_m^2/2 \\ &\quad -(\lfloor (B_{A_m+c+\epsilon_{m2}}+c+\epsilon_{m1})\phi\rfloor - B_{A_n+c+\epsilon_{n2}}); \\ \xi_n &= -(\lfloor n\phi\rfloor - n\phi)(2\phi-3)+\epsilon_{n2}(5-3\phi)+\epsilon_n(3-2\phi)/2-\epsilon_n^2/2 \\ &\quad -(\lfloor (B_{A_n+c+\epsilon_{n2}}+c+\epsilon_{n1})\phi\rfloor - B_{A_n+c+\epsilon_{n2}}). \end{aligned}$$

Then $\xi=\xi_m-\xi_n$, and $|\xi_m|,|\xi_n|<(2\phi-3)+(5-3\phi)+(2\phi-3)/2+1/2=1$. which completes the proof. □

Corollary 4.3. $|\alpha_{B_{A_m+c+\epsilon_{m2}}+c+\epsilon_{m1}}-\alpha_{B_{A_n+c+\epsilon_{n2}}+c+\epsilon_{n1}}|\leq \max(|\alpha_m-\alpha_n|-1,2)$, *when* $-1\leq \epsilon_{m1},\epsilon_{n1}\leq 1$ *and* $0\leq\epsilon_{m2},\epsilon_{n2}\leq 1$,

Proof. By Lemma 4.2, $|\alpha_{B_{A_m+c+\epsilon_{m2}}+c+\epsilon_{m1}}-\alpha_{B_{A_n+c+\epsilon_{n2}}+c+\epsilon_{n1}}|<|\alpha_m-\alpha_n|(2\phi-3)+2$.

Since α_n is always an integer,

$$\begin{aligned} & |\alpha_{B_{A_m+c+\epsilon_{m2}}+c+\epsilon_{m1}}-\alpha_{B_{A_n+c+\epsilon_{n2}}+c+\epsilon_{n1}}| \\ \leq\ & \lfloor |\alpha_m-\alpha_n|(2\phi-3)+2\rfloor \\ \leq\ & \max(|\alpha_m-\alpha_n|-1,2). \end{aligned}$$

□

Lemma 4.4. *α_n is bounded for all n.*

Proof. As we discussed at the beginning of the section, any $n\geq B_{A_{n_0}+c}$ can be written as $B_{A_m+c+\epsilon_2}+c+\epsilon_1$, where $\epsilon_1\in\{-1,0,1\}$ and $\epsilon_2\in\{0,1\}$. So for any n large enough, we can construct a sequence $a_1,\ldots,a_k$ such that $a_k=n$; $a_i=B_{A_{i-1}+c+\epsilon_2^{(i-1)}}+c+\epsilon_1^{(i-1)}$,; and $A_{n_0}\leq a_1<B_{A_{n_0}+c+1}+c+1$, where $\epsilon_1^{(i-1)}\in\{-1,0,1\}$ and $\epsilon_2^{(i-1)}\in\{0,1\}$ for $1<i\leq k$.

Using the same notation as in Lemma 4.2, we know

$$\alpha_{a_i}-\alpha_{a_{i-1}}=(\alpha_{a_{i-1}}-\alpha_{a_{i-2}})(3-2\phi)+\xi_i-\xi_{i-1},$$

where ξ_i is determined solely by the values of i.

Define $\delta_i=\sum_{j=0}^{i}(3-2\phi)^j$, it is easy to see that $\delta_0=1$, $|\delta_i|\leq 1$ and $\delta_i=1+(3-2\phi)\delta_{i-1}$.

$$
\begin{aligned}
\text{Now}\quad & \alpha_m \\
= \quad & \alpha_{a_1} + \textstyle\sum_{i=2}^{k}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \alpha_{a_1} + (\alpha_{a_{k-1}} - \alpha_{a_{k-2}})(3-2\phi) + \xi_k - \xi_{k-1} + \textstyle\sum_{i=2}^{k-1}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \alpha_{a_1} + (\xi_k - \xi_{k-1})\delta_0 + (\alpha_{a_{k-1}} - \alpha_{a_{k-2}})\delta_1 + \textstyle\sum_{i=2}^{k-2}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \alpha_{a_1} + (\xi_k - \xi_{k-1})\delta_0 + (\xi_{k-1} - \xi_{k-2})\delta_1 \\
& + (\alpha_{a_{k-2}} - \alpha_{a_{k-3}})\delta_2 + \textstyle\sum_{i=2}^{k-3}(\alpha_{a_i} - \alpha_{a_{i-1}}) \\
= \quad & \cdots \\
= \quad & \alpha_{a_1} + \textstyle\sum_{i=3}^{k}(\xi_i - \xi_{i-1})\delta_{k-i} + (\alpha_{a_2} + \alpha_{a_1})\delta_{k-2}.
\end{aligned}
$$

Since $A_{n_0} \leq a_1 < B_{A_{n_0}+c+1} + c + 1$, there are only finitely many choices of a_1, so the first term in the last equation is bounded. Likewise, the third term is also bounded. So we only have to inspect the second term.

$$
\begin{aligned}
& |\textstyle\sum_{i=3}^{k}(\xi_i - \xi_{i-1})\delta_{k-i}| \\
= \quad & |\textstyle\sum_{i=3}^{k}\xi_i\delta_{k-i} - \sum_{i=3}^{k}\xi_{i-1}\delta_{k-i}| \\
= \quad & |\textstyle\sum_{i=3}^{k}\xi_i\delta_{k-i} - \sum_{i=2}^{k-1}\xi_i\delta_{k-i-1}| \\
= \quad & |\textstyle\sum_{i=3}^{k-1}\xi_i(\delta_{k-i} - \delta_{k-i-1}) + \xi_k\delta_0 - \xi_2\delta_{k-3}| \\
\leq \quad & |\textstyle\sum_{i=3}^{k-1}\xi_i(\delta_{k-i} - \delta_{k-i-1})| + |\xi_k\delta_0| + |\xi_2\delta_{k-3}| \quad \text{which completes the proof.} \\
= \quad & |\textstyle\sum_{i=3}^{k-1}\xi_i(3-2\phi)^{k-i}| + 1 + 1 \\
\leq \quad & \textstyle\sum_{i=3}^{k-1}|\xi_i||3-2\phi|^{k-i} + 1 + 1 \\
< \quad & \textstyle\sum_{i=0}^{\infty}|3-2\phi|^{i} + 1 + 1 \\
< \quad & 4,
\end{aligned}
$$

□

Using the same method, we can investigate the behavior of the sequence $\{\alpha_m\}_{m\geq n_0}$:

Let $\beta_3 = (\lfloor A_m\phi\rfloor - \lfloor A_n\phi\rfloor) - (A_m - A_n)\phi$, and $\beta_4 = (\lfloor m\phi\rfloor - \lfloor n\phi\rfloor) - (m-n)\phi$.

$$
\begin{aligned}
& \alpha_{A_m} - \alpha_{A_n} \\
= \quad & A_{A_m} - A_{A_n} - (\lfloor A_m\phi\rfloor - \lfloor A_n\phi\rfloor) \\
= \quad & B_m - B_n - (A_m - A_n)\phi - \beta_3 \\
= \quad & (A_m - A_n)(1-\phi) + (m-n) - \beta_3 \\
= \quad & (\lfloor m\phi\rfloor - \lfloor n\phi\rfloor + \alpha_m - \alpha_n)(1-\phi) + (m-n) - \beta_3
\end{aligned}
$$

$$= \quad ((m-n)\phi + \beta_4 + \alpha_m - \alpha_n)(1-\phi) + (m-n) - \beta_3$$
$$= \quad -(\alpha_m - \alpha_n)(\phi - 1) - \beta_3 - \beta_4(\phi - 1).$$

From the proof of Lemma 4.4, we can see that when $|\alpha_m - \alpha_n| \geq 3$, $(\alpha_{A_m} - \alpha_{A_n})$ and $(\alpha_m - \alpha_n)$ always have different signs. $(\lfloor (m+1)\phi \rfloor - \lfloor m\phi \rfloor)| \leq 1$, Let us consider $\alpha(m) = \alpha_m$ as a function. The graph of the function is a set of discrete points that oscillate. The amplitude of graph, if we are allowed to abuse the word, decreases slowly but persistently as m grows. By Theorem 4.1, the amplitude eventually decreases to 1, when the oscillation of the graph becomes somewhat unpredictable.

Lemma 4.5. *In the two conjectures on the N-heap Wythoff's game,*

$$A_n^{N-1} = \text{mex}(\{A_i^{N-1}, A_i^N :\ 0 \leq i < n\} \ \cup \ T), \tag{4.1}$$

where T is a finite set depending only on $A^1, \ldots, A^{N-2}$. In fact, $T = \{a : \exists b$ and k, such that $A^{k-1} \leq b \leq A^k$ and $(A^1, \ldots, A^{k-1}, b, A^k, \ldots, A^{N-2}$, $a)$ is a P-position$\}$.

Proof. By definition, $T = \mathbb{Z}_{\geq 0} - \{A_i^{N-1}, A_i^N :\ i > 0\}$. Write T' as the last set in the lemma, and we claim $T = T'$.

First to prove $T' \subset T$, we want to show that for any $a \in T'$, $(A^1, \ldots, A^{N-2}, a, b)$ is an N-position for all $b \geq A^{N-2}$. This is true because by the definition of T', we can always remove tokens from the last pile to create a P-position.

Secondly, given $a \in T$, $(A^1, \ldots, A^{N-2}, a, b)$ is an N-position for any $b \geq a$ by the definition of T. There are several kind of moves from this position to find a P-position:

(1) Remove $a_1, \ldots, a_N$ tokens from all corresponding piles, where $a_1 \oplus \cdots \oplus a_N = 0$, so that $(A^1 - a_1, \ldots, A^{N-2} - a_{N-2}, a - a_{N-1}, b - a_N)$ is a P-position.
(2) Remove $a_k \leq A^k$ tokens from the k-th pile, so that $(A^1, \ldots, A^{k-1}, A^k - a_k$, $A^{k+1}, \ldots, A^{N-2}, a, b)$ is a P-position;
(3) Remove $a_{N-1} \leq a$ tokens from the $(N-1)$-th pile, so that $(A^1, \ldots, A^{N-2}, a - a_{N-1}, b)$ is a P-position;
(4) Remove $a_N \leq b$ tokens from the N-th pile, so that $(A^1, \ldots, A^{N-2}, a, b - a_N)$ is a P-position;

There are only finitely many possible moves using the first three kinds of moves, but there are infinitely many choices of b. So there are cases for the fourth kind of move, i.e., there exists an integer b_1, b_2 such that $(A^1, \ldots, A^{N-2}, a, b_1 - b_2)$ is a P-position. Again by the definition of T, we must have $b_1 - b_2 \leq A^{N-2}$. Otherwise, $a \in \{A^{N-1}\}$. Therefore $a \in T'$, thus $T \subset T'$.

Since there are only finitely many choices of $b = b_1 - b_2$ ($\sum_{i=1}^{N-2}(A^i+1)$ to be exact), and each choices of b can yield at most one corresponding a as in the definition of T', $T = T'$ must be finite.

To prove equation 4.1, let $a = \text{mex}(\{A_i^{N-1}, A_i^N : 0 \le i < n\} \cup T)$. $A_n^{N-1} \ge a$ because a is the smallest integer available. Supposedly $A_n^{N-1} \ne a$, since we assume $A_{i-1}^{N-1} < A_i^{N-1}$ for all i, so a did not appear in $\{A_i^{N-1}\}_{i<n}$ and can no longer appear in $\{A_i^{N-1}\}_{i \ge n}$, thus a cannot be in $\{A_i^{N-1}\}$. Similarly since we also assume $A_i^{N-1} \le A_i^N$ for all i, a cannot be in $\{A_i^N\}$ either. Therefore $a \in \mathbb{Z}_{\ge 0} - \{A_i^{N-1}, A_i^N : 0 \le i < n\} = T$, but this is contrary to the definition of a. □

Corollary 4.6. *The first of Fraenkel's two conjectures on the N-heap Wythoff's game implies the second.*

Proof. Conjecture 1, together with the previous lemma, states that the P-positions for any given m form a Wythoff's sequence, while Conjecture 2 states that it satisfies some properties of a special Wythoff's sequence. The result follows directly from Theorem 4.1. □

Theorem 4.7. *Given a Wythoff's sequence $\{(A_n, B_n)\}_{n \ge n_0}$ and α are as in Definition 2, $\alpha = -c$.*

Proof. Let $\beta_5 = \lfloor (A_n + c)\phi \rfloor - (A_n + c)\phi$ and $\beta_6 = \lfloor n\phi \rfloor - n\phi$. Then

$$A_n + n - 1 = B_n - 1 = A_{A_n+c} = \lfloor (A_n + c)\phi \rfloor + \alpha + \epsilon_{A_n+c} = A_n\phi + c\phi + \alpha + \epsilon_{A_n+c} + \beta_5.$$

So

$$\begin{aligned} \epsilon_{A_n+c} + \beta_5 &= A_n(1-\phi) - 1 + n - c\phi - \alpha \\ &= (n\phi + \alpha + \epsilon_n + \beta_6)(1-\phi) + n - c\phi - \alpha - 1 \\ &= -(c+\alpha)\phi + (\epsilon_n + \beta_6)(1-\phi) - 1, \end{aligned}$$

hence

$$-(c+\alpha)\phi = \epsilon_{A_n+c} + \beta_5 + (\epsilon_n + \beta_6)(\phi - 1) + 1.$$

Note that the left-hand side of the last equation does not depend on the choice of n, while the right-hand side does. Since both c and α are integers, the theorem is proved if we can make the right choice of n so that the absolute value of the right-hand side is less than ϕ. Because $-1 < \beta_5, \beta_6 < 0$, we have

$$\epsilon_{A_n+c} + \epsilon_n(\phi - 1) + 1 - \phi < -(c+\alpha)\phi < \epsilon_{A_n+c} + \epsilon_n(\phi - 1) + 1. \tag{4.2}$$

We also know that $\epsilon_{A_n+c}, \epsilon_n \in \{0, \pm 1\}$, therefore the proof is completed if we can find an integer N such that

$$\epsilon_{A_N+c} = 0; \quad \text{or} \tag{4.3}$$

$$\epsilon_N = 0 \text{ and } \epsilon_{A_N+c} \in \{0, -1\}; \quad \text{or} \tag{4.4}$$

$$\epsilon_N = -1 \text{ and } \epsilon_{A_N+c} \in \{0, 1\}. \tag{4.5}$$

The reader can easily check that in any of the conditions above, the absolute value of either end of the inequality 4.2 is at most ϕ.

First we can assume ϵ_n is not a constant, otherwise we can adjust the value of α so that ϵ_n is always 0. That will satisfy condition 4.3 automatically. Secondly, notice that $-1 = 1 - 2 < (A_n - A_{n-1}) - (\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor) < 2 - 1 = 1$. Since $|\epsilon_n| \le 1$ and $|\epsilon_n - \epsilon_{n-1}| = |(A_n - A_{n-1}) - (\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor)| \le 1$, there always exists an n large enough so that $\epsilon_n = 0$. By the condition 4.4 above, we only have to consider the case when

$$\epsilon_n = 0 \quad \text{and } \epsilon_{A_n+c} = 1. \tag{4.6}$$

From now on, we always assume n is large enough.

There are two possibilities for $A_n - A_{n-1}$:

If $A_n = A_{n-1} + 1$, by Corollary 3.2, there exist m such that $n = B_m + 1 + c$; and by Lemma 2.1, there exists m' such that $B_m + 1 = A_{m'}$. Therefore, $\epsilon_{A_{m'}+c} = \epsilon_n = 0$, which proves the theorem by choosing $N = m'$ and using condition 4.3.

If $A_n = A_{n-1} + 2$, then

$$3 \le \quad \lfloor (A_n + c)\phi \rfloor - \lfloor (A_{n-1} + c)\phi \rfloor \quad \le 4. \tag{4.7}$$

There also exists m such that $A_{n-1} + 1 = B_m = A_{A_m+c} + 1$, i.e.,

$$n - 1 = A_m + c. \tag{4.8}$$

Furthermore $A_{A_n+c} - A_{A_{n-1}+c} = (A_{B_m+1+c} - A_{B_m+c}) + (A_{B_m+c} - A_{B_m-1+c}) = 3$ by Corollary 3.2, which means $\lfloor (A_n + c)\phi \rfloor - \lfloor (A_{n-1} + c)\phi \rfloor + \epsilon_{A_n+c} - \epsilon_{A_{n-1}+c} = 3$. Because of the fact that $|\epsilon_{A_{n-1}+c}| < 1$ and the ones listed in 4.6 and 4.7,

$$\lfloor (A_n + c)\phi \rfloor - \lfloor (A_{n-1} + c)\phi \rfloor = 3 \text{ and } \epsilon_{A_{n-1}+c} = 1.$$

Since $2 = A_n - A_{n-1} = \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor - \epsilon_{n-1}$, we have

- either $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 1$ and $\epsilon_{n-1} = -1$,
- or $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 2$ and $\epsilon_{n-1} = 0$.

In the former case we can prove the theorem by choosing $N = n - 1$ and using condition 4.5 because $\epsilon_{n-1} = -1$ and $\epsilon_{A_{n-1}+c} = 1$; while in the latter case $\epsilon_{A_m+c} = \epsilon_{n-1} = 0$, so we can choose $N = m$ and use condition 4.3.

Thus we have completed the proof. □

Theorem 3.1, together with the comments at the end of the Section 3, indicates that any Wythoff's sequence is "shifted" Wythoff's pairs. It also maintains the relationship with the golden section with another "shift" α and some "controlled error" ϵ. Theorem 4.7 tells us the values of the two shifts are in fact the same.

Example 4. Given any integer a, consider the sequence $\{(A_n = \lfloor n\phi \rfloor + a, B_n = \lfloor n\phi \rfloor + n + a)\}$, with n large enough. Since it is created from the Wythoff's pairs with a simple shift a, the sequence is a special Wythoff's sequence with $\alpha = a$ and $\epsilon_n \equiv 0$. At the mean time, $A_{A_n - a} = A_{\lfloor n\phi \rfloor} = \lfloor \lfloor n\phi \rfloor \phi \rfloor + a = \lfloor n\phi \rfloor + n - 1 + a = B_n - 1$, where the equation in the middle can be derived from the fact that the constant c for the Wythoff's pairs is 0, or from [1]. Similarly, $A_{B_n - a} = A_{\lfloor n\phi \rfloor + n} = \lfloor (\lfloor n\phi \rfloor + n)\phi \rfloor + a = 2\lfloor n\phi \rfloor + n + a = A_n + B_n - a$. So the constant c for the sequence is $-a = -\alpha$.

Example 5. Let us go back to the Wythoff's sequence shown in Example 1. After the 8-th pair, we can find out that $\alpha_n = A_n - \lfloor n\phi \rfloor$ ranges from -2 to -4, so $\alpha = -3 = -c$.

The implication of Theorem 4.7 follows. To determine the value of α for any Wythoff's sequence, especially those constructed using the method in Theorem 2.2, we usually need to calculate a large number of pairs as required. However based on Theorem 4.7, we only need the pairs at the beginning of the sequence. As shown in the proof of Theorem 3.1, all we need to know is the integer k such that $A_k = B_{n_0} - 1$, which is to find all the A's less than B_{n_0}. So by using the notation in the proof of Corollary 3.3, $f(B_{n_0}) = A_{B_{n_0}+c} - B_{n_0} - n_0 + 1 = A_{n_0} - n_0 + 1 + c = B_{n_0} - 2n_0 + 1 + c$, therefore it only requires the values of roughly $B_{n_0} - 2n_0 + 1$ pairs of integers.

Example 6. If $S_1 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 18, 22, 24, 25, 50, 52, 55, 69, 80, 86, 90, 94, 101, 103, 104\}$ and $S_2 = \{0, 1, 2, 3, 4, 5, 6, 8, 12, 14, 19, 22, 35, 44, 45, 46, 51, 54, 56\}$ as in Theorem 2.2, the result follows:

- Starting from the 50-th pair, the result becomes a Wythoff's sequence;
- Starting from the 470-th pair, α_n stabilizes within the range of $[-7, -9]$;
- However, based on Theorem 3.1 and Theorem 4.7, we only need to calculate up to 64 pairs to know c, and thus α.

The exact list of the sequence is omitted.

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