Math 603, Spring 2007, Homework Set 8 Key

Due Friday, March 30th
Total points: 50 points

1.[10 points] Let $\{X_n\}_{n\geq 0}$ be a Markov chain on Σ, where $|\Sigma| < \infty$. Prove that no null recurrent states exist (**Hint**: Use the decomposition of Σ into transient states, and recurrent states).
Solution: By our result on the partitioning of the state space, because the state space is finite, there exists a $d \in \mathbb{Z}^+$ such that $\Sigma = \Sigma_T \cup C_1 \cup \cdots \cup C_d$, where Σ_T is the set of transient states, and C_i is a closed, irreducible set of recurrent states, $i = 1, \ldots, d$. Since $C_i \subset \Sigma$, $|C_i| \leq |\Sigma| < \infty$, so C_i is finite, it has at least on positive recurrent state, and since it is irreducible, all states in C_i are positive recurrent, $i = 1, \ldots, d$. Therefore, there exist no null recurrent states.

2. Let π_0 and π_1 be two different stationary distributions for P, that is there exists at least one $\omega \in \Sigma$ such that $\pi_0(\omega) \neq \pi_1(\omega)$.
a).[5 points] Show that for any $p \in [0, 1]$, $\pi_p = p\pi_1 + (1-p)\pi_0$ is also a stationary distribution for P. Note, $\pi_p(x) = p\pi_1(x) + (1-p)\pi_0(x)$ for all $x \in \Sigma$.
Solution:

$$\sum_{x\in\Sigma} \pi_p(x)P(x, y) = \sum_{x\in\Sigma} p\pi_1(x)P(x, y) + (1-p)\pi_0(x)P(x, y)$$
$$p\sum_{x\in\Sigma} \pi_1(x)P(x, y) + (1-p)\sum_{x\in\Sigma} \pi_0(x)P(x, y) = p\pi_1(y) + (1-p)\pi_0(y) = \pi_p(y).$$

b).[5 points] Show that $\pi_{p_1} = \pi_{p_2} \iff p_1 = p_2$ for $p_1, p_2 \in [0, 1]$.
Solution: Only the necessity must be shown. As π_1 and π_0 are distinct, assume that for $\omega \in \Sigma$, $\pi_0(\omega) \neq \pi_1(\omega)$. Then, since $\pi_{p_1} = \pi_{p_2}$,

$$p_1\pi_1(\omega) + (1-p_1)\pi_0(\omega) = p_2\pi_1(\omega) + (1-p_2)\pi_0(\omega)$$
$$\Rightarrow p_1(\pi_1(\omega) - \pi_0(\omega)) = p_2(\pi_1(\omega) - \pi_0(\omega)) \Rightarrow p_1 = p_2$$

c).[5 points] Let x and y be two positive recurrent states, and assume that $x \not\rightarrow y$. Show that an infinite number of stationary distributions exist(**Hint**: Use the decomposition of the state space again).
Solution: Recall that there exists a $d \in \mathbb{Z}^+ \cup \{\infty\}$ such that $\Sigma = \Sigma_T \cup C_1 \cup \cdots \cup C_d$, where Σ_T is the set of transient states, and C_i is a closed, irreducible set of recurrent states, $i = 1, \ldots, d$. Therefore, because x and y are both recurrent, but do not communicate, there exists $i \neq j \in \mathbb{Z}^+$ such that $x \in C_i$, $y \in C_j$. Now, as x and y are both positive recurrent, and all sets C_k are irreducible, $k \geq 1$, it follows that C_i and C_j. Therefore, when the chain is in

C_i or C_j, it behaves like a positive recurrent chain on either of these respective state spaces. Thus, there are two stationary distributions π_i and π_j such that $\pi_i(C_i) = 1$, $\pi_j(C_j) = 1$. As these are distinct distributions, the results follows from part a, with $\pi_0 = \pi_i$ and $\pi_1 = \pi_j$, and p ranging in $[0, 1]$.

3. A professor has two light bulbs in his garage. Let X_n be the number of light bulbs working at the end of the nth day. Bulbs are replaced only when both bulbs are burned out, and the replacement occurs the next morning. When both are working at the beginning of a day, one of the two will go out with probability .02 sometime that day, and we assume that there is no probability that both will go out that day. However, when only one is there at the beginning of the day, it will burn out with probability .05.
Preliminary Work: Let 0 = no bulbs working, 1 = 1 bulb working, 2 = 2 bulbs working. Let X_n = # of working bulbs at the end of the nth day. $\{X_n\}_{n\geq 0}$ is a Markov chain with $P(X_{n+1} = 0|X_n = 0) = 0$, as one begins with two working bulbs, and we eliminate the possibility of two light bulbs burning out in the same day. $P(X_{n+1} = 1|X_n = 0) = .02$, implying $P(X_{n+1} = 2|X_n = 0) = .98$. The way the dynamics are described, $P(X_{n+1} = i|X_n = 2) = P(X_{n+1} = i|X_n = 0)$, $i = 0, 1, 2$. Finally, $P(X_n = 2|X_{n-1} = 1) = 0$, since you have to wait until both bulbs are burned out to replace the bulbs. By description $P(X_{n+1} = 0|X_n = 1) = .05$, implying $P(X_{n+1} = 1|X_n = 1) = .95$. This gives the following transition matrix

$$P = \begin{bmatrix} 0 & .02 & .98 \\ .05 & .95 & 0 \\ 0 & .02 & .98 \end{bmatrix}.$$

i).[5 points] What is the long-run fraction of time that there is exactly one bulb working?
Solution: From the form of P, X_n is an irreducible, positive recurrent Markov chain. Thus, there exists a stationary distribution π, $\pi^t P = \pi^t$, and $\pi(x) > 0$ for all $x \in \Sigma$. π may thus be found by finding a left eigenvector of P with eigenvalue 1, and normalizing the eigenvector. We find this eigenvector by solving $x^t P = x^t$, written as the following system of equations:

$$\begin{aligned} 0 + .05x_1 + 0 &= x_0 \\ .02x_0 + .95x_1 + .02x_2 &= x_1 \\ .98x_0 + 0 + .98x_2 &= x_2 \end{aligned}$$

Since this isn't a set of linearly independent equations, we can choose one x_i as we like. Choose $x_1 = 1$, and this then, after solving, this leads to $x_0 = .05$, $x_1 = 1$, $x_2 = 2.45$, which gives that

$$\pi(0) = \frac{.05}{3.5} = \frac{1}{70},\ \pi(1) = \frac{1}{3.5} = \frac{20}{70},\ \pi(2) = \frac{2.45}{3.5} = \frac{49}{70}.$$

Then, irreducibility and positive recurrence imply that

$$\frac{1}{n}\sum_{k=1}^{n} 1_1(X_k) = \frac{N_n(1)}{n} \to \pi(1) = \frac{20}{70} \text{ almost surely}$$

is the long run proportion of time there is exactly one light bulb working.

ii).[5 points] What is the expected time between light bulb replacements.

Solution: The expected time between light bulb replacements is the expected first return time to 0, given you start in 0, which is $E_0(T_0) = m_0 = \frac{1}{\pi(0)} = 70$.

iii).[5 points] What is the long-run probability there are exactly two bulbs bulbs working.

Hint: Use the Ergodic Theorem, and theorem from class regarding $\frac{N_n(y)}{n}$.

Solution: This is simply $\lim_n P_\tau(X_n = 2) = \pi(2) = \frac{49}{70}$.

4. Let $X_n,\ n \geq 0$, be the Ehrenfest chain, which is a Birth and Death chain on $\{0, 1, \ldots, N\}$ such that

$$\begin{aligned} P(x, x-1) &= \frac{x}{N},\ 1 \leq x \leq N \\ P(x, x+1) &= \frac{N-x}{N},\ 0 \leq x \leq N-1. \end{aligned}$$

Assume that $N = 4$ and $X_0 = 0$.

Preliminary Work: The Ehrenfest chain with $N = 4$, $X_0 = 0$ gives the following Markov chain: the initial distribution is $1_0(\cdot) : \Sigma \to [0,1]$,and transition kernel

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{2}{4} & 0 & \frac{2}{4} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Obviously, the Ehrenfest chain is a finite, irreducible Birth and Death chain that must have a stationary distribution. Recall,

$$L_x = \begin{cases} 1 & : \quad x = 0 \\ \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} & : \quad x \in \{1, 2, 3, 4\}. \end{cases}$$

Thus, $L_0 = 1$, and

$$L_1 = \frac{1}{\frac{1}{4}},\ L_2 = \frac{1 \times \frac{3}{4}}{\frac{1}{4} \times \frac{2}{4}},\ L_3 = \frac{1 \times \frac{3}{4} \times \frac{2}{4}}{\frac{1}{4} \times \frac{2}{4} \times \frac{3}{4}},\ L_4 = \frac{1 \times \frac{3}{4} \times \frac{2}{4} \times \frac{1}{4}}{\frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} \times 1},$$

This implies that the stationary distribution π of this Ehrenfest chain is given by

$$\pi(0) = \frac{1}{16}, \pi(1) = \frac{1}{4}, \pi(2) = \frac{3}{8}, \pi(3) = \frac{1}{4}, \pi(4) = \frac{1}{16}.$$

Now, since the chain is finite, and irreducible it is positive recurrent, and therefore, the Ergodic Theorem holds. In this case, we see that this Ehrenfest chain has period 2, because you can only move either up or down 1 step. Thus,

$$\begin{aligned} P^{2n}(x,y) &\to 2\pi(y) \text{ if } x,y \text{ both even or both odd,} \\ P^{2n+1}(x,y) &= 0 \text{ for all } n, \text{ if } x,y \text{ both even or both odd,} \\ P^{2n}(x,y) &= 0 \text{ for all } n, \text{ if } x,y \text{ one is odd, one is even,} \\ P^{2n+1}(x,y) &\to 2\pi(y) \text{ if } x,y \text{ one is odd, one is even.} \end{aligned}$$

a).[5 points] Find the approximate distribution of X_n for n large and even.
Solution: $P_0(X_n = 0) = P^{2(\frac{n}{2})}(0,0) = 2\pi(0) = \frac{2}{16} = \frac{1}{8}$ as 0 and 0 are both even, and n is assumed even. Thus, with n even and large,

$$\begin{aligned} P_0(X_n = 1) &= P^{2(\frac{n}{2})}(0,1) = 0 \text{ as 0 is even, 1 is odd,} \\ P_0(X_n = 2) &= P^{2(\frac{n}{2})}(0,2) = 2\pi(2) = \frac{3}{4} \text{ as 0 is even, 2 is even,} \\ P_0(X_n = 3) &= P^{2(\frac{n}{2})}(0,3) = 0 \text{ as 0 is even, 3 is odd,} \\ P_0(X_n = 4) &= P^{2(\frac{n}{2})}(0,4) = 2\pi(4) = \frac{1}{8} \text{ as 0 is even, 4 is even.} \end{aligned}$$

b).[5 points] Find the approximate distribution of X_n for n large and odd.
Solution: Similarly, as n is now assumed odd and large,

$$\begin{aligned} P_0(X_n = 0) &= P^{2(\frac{n}{2})}(0,0) = 0 \text{ as 0 is even, 0 is even,} \\ P_0(X_n = 1) &= P^{2(\frac{n}{2})}(0,1) = 2\pi(1) = \frac{1}{2} \text{ as 0 is even, 1 is odd,} \\ P_0(X_n = 2) &= P^{2(\frac{n}{2})}(0,2) = 0 \text{ as 0 is even, 2 is even,} \\ P_0(X_n = 3) &= P^{2(\frac{n}{2})}(0,3) = 2\pi(3) = \frac{1}{2} \text{ as 0 is even, 3 is odd,} \\ P_0(X_n = 4) &= P^{2(\frac{n}{2})}(0,4) = 0 \text{ as 0 is even, 4 is even.} \end{aligned}$$